# Deformation Theory and Operads

Gabriel C. Drummond-Cole

July 9, 2012

Recall that for $V$ a vector space over $k$, we had $TV = \bigoplus V^{\otimes n}$ and $LV = \bigcap_{V \subset g \subset TV} g$. Then

$$\tilde{\Delta}(v_1, \ldots, v_n) = \sum_{p,q\text{-shuffles},\ p+q=n} (v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \otimes (v_{\sigma(p+1)}, \ldots, v_{\sigma(n)})$$

So $\gamma : TV \to LV$ takes 1 to 0 and $v \in V$ to itself. It takes $v_1, \ldots, v_n$ to $[v_1, [v_2, \ldots, [v_{n-1}, v_n] \ldots]]$

**Proposition 1** *For $x \in V^{\otimes n} \subset TV$ the following are equivalent:*

1. $x \in LV$
2. $\tilde{\Delta}(x) = x \otimes 1 + 1 \otimes x$
3. $\gamma(x) = n.x$

The easiest implication is the third to the first. Assume that $\gamma(x) = n.x$. Then $x = \frac{\gamma(x)}{n}$ so it's in the free Lie algebra, since $\gamma$ is a map to the free Lie algebra. All iterated brackets are in the free Lie algebra.

Let me show the first implies the second. Let $\tilde{g}$ be those $x$ in $TV$ such that $\tilde{\Delta}(x) = x \otimes 1 + 1 \otimes x$. I claim that $\tilde{g}$ is a Lie subalgebra of $TV$. If I have $x, y \in \tilde{g}$, and I take their bracket, I take $\tilde{\Delta}([x, y])$, what is that? Let's see. It's $\tilde{\Delta}(xy - yx)$. Recall that $\tilde{\Delta}$ is an algebra map, so this is

$$\tilde{\Delta}(x)\tilde{\Delta}(y) - \tilde{\Delta}(y)\tilde{\Delta}(x) = (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x)$$

If you foil this, you get

$$[x, y] \otimes 1 + 1 \otimes [x, y]$$

. So this is a subalgebra of $TX$ as a Lie algebra. So $LV \subset \tilde{g}$.

The last implication assumes that $\tilde{\Delta}(x) = x \otimes 1 + 1 \otimes x$. We will see in a moment that $n.x = \bullet \circ (\gamma \otimes id_T V) \circ \tilde{\Delta}$. This is a map from $TV$ to $TV \otimes TV$ to $LV \otimes TV$ to $TV$. Then this is $\bullet(\gamma \otimes id)(x \otimes 1 + 1 \otimes x)$. Since $\gamma(1) = 0$ we have $\bullet(\gamma(x) \otimes 1) = \gamma(x)$. The only thing I owe you is to show that $nx = \bullet(\gamma \otimes id)\tilde{\Delta}$. We show this by induction on $n$. The first couple of pieces are pretty trivial. If $n = 0$ this is a scalar and will eventually go to 0. If you start with $v \in V$ you get $\bullet(\gamma(v) \otimes 1) = \gamma(v)$.

Let $n \geq 2$; we'll do a little induction here. Assume that, I'm going to show it for homogeneous elements. Let $x = (v_1, \ldots, v_n)$. Write $\tilde{\Delta}$, apply it to $(v_2, \ldots, v_n)$, so this is $1 \otimes (v_2 \ldots v_n) + \sum_i a_i \otimes b_i$. The only thing I do is that $a_i \in V^{\otimes \geq 1}$.

By induction, $(n-1)(v_2, \ldots, v_n)$ is $\bullet(\gamma \otimes id)\tilde{\Delta}(v_2, \ldots, v_n) = \bullet(\gamma \otimes id)(1 \otimes (v_2, \ldots, v_n) + \sum a_i \otimes b_i] = \sum \gamma_i(a_i), b_i)$.

Then

$$\bullet(\gamma\otimes id)\tilde{\Delta}(v_1, \ldots, v_n) = \bullet(\gamma\otimes id)[v_1\otimes 1+1\otimes v_1)(1\otimes v_2 \ldots v_n)] = v_1\otimes v_2 \ldots, v_n+\sum v_i a_i\otimes b_i+1 \ldots v_1 \ldots v_n+\sum a_i\otimes v$$

$$= \bullet(v_1 \otimes v_2, \ldots, v_n) + \sum \gamma(v_1 a_i)b_i + \sum(\gamma(a_i))v_i b_i$$

The last piece of this is $[v_1, \gamma(a_i)] - v_1\gamma(a_i) - \gamma(a_i)v_1$ so we get

$$v_1 v_2 \ldots, v_n + \sum v_1\gamma(a_i)b_i = v_1 \ldots v_n + (n-1)v_1 \ldots v_n = n(v_1 \ldots v_n)$$

**Corollary 1** *for all $\theta : V \to g$ there exists a unique map $\tilde{\theta} : LV \to g$ a Lie algebra map which extends:*

$$\begin{tikzcd} & LV \arrow[d, "\tilde{\theta}"] \\ V \arrow[ur, "inc"] \arrow[r, "\theta"'] & g \end{tikzcd}$$

*Proof.* Define $\tilde{\theta}([v_1, [v_2, \ldots, [v_{n-1}, v_n] \ldots]])$ to be $\{\theta v_1, \{\theta v_2, \ldots \{\theta v_{n-1}, \theta v_n\} \ldots\}\}$ be the unique map induced by the Lie relation. We need to show that for $x, y$ in $LV$, we have $\tilde{\theta}([x, y]) = \{\tilde{\theta}x, \tilde{\theta}y\}$ By induction on $x \in L^n V$ we see for $n = 1$ that $x \in V$. For $n > 1$ let $x = [v, \bar{x}]$ Then

$$\tilde{\theta}([x, y]) = \tilde{\theta}([[v, \bar{x}], y]) \underbrace{=}_{\text{Jacobi}} \tilde{\theta}(\pm[v, [\bar{x}, y]] \pm [\bar{x}, [v, y]]) \underbrace{=}_{\text{induction}} \pm\{\tilde{\theta}v, \tilde{\theta}[\bar{x}, y]\} \pm \{\tilde{\theta}\bar{x}, \tilde{\theta}[v, y]\} \underbrace{=}_{\text{induction}} \pm\{\tilde{\theta}v, \tilde{\theta}\{\tilde{\theta}\bar{x}, \tilde{\theta}y\}\}\pm$$

Back to the Lie operad. $Lie(n)$ is the span of monomials in $LV$ with exactly one $v_1, \ldots, v_n$

**Lemma 1** *The brackets $[v_{\sigma(1)}, [v_{\sigma(2)}, \ldots, [v_{\sigma(n-1)}, v_n], \ldots, ]]$ for $\sigma \in \Sigma_{n-1}$ form a basis for $Lie(n)$. In particular, the dimension is $(n-1)!$*

You just use anticommutativity to put one thing at the end, and then Jacobi to move the other brackets to this form.

[Discussion of $Lie \to Ass \to Com$. We agree that the maps don't go in the other direction.]

Let's to another example, the little disks operad. Let $\mathscr{C}$ be the topological category, and let $D^k$ be $\{x \in \mathbb{R}^k | \, |x| \leq 1\}$ be the $k$-ball. We define a little disk to be a map $f : D^k \to D^k$ of the form $f(x_1, \ldots, x_k) = y_1 + cx_1, \ldots, y_k + cx_k$ so that $c \leq 1$ and $f(D^k) \subset D^k$. The little $k$-disk operad consists of $D_k(n) = \{f_1, \ldots, f_n\}$ where each $f_i$ is a little disk such that their images can touch but not intersect in the interiors.

[What if they can't touch?] I think something breaks but I don't know what.

Let me draw a picture. [picture]

This sits in $\mathbb{R}^{n(k+1)}$. Now I need to tell you the structures. The identity in $D_k(1)$ is the identity of $D^k$. I'm using the cartesian product, so $(f_1, \ldots, f_n).\sigma = (f_\sigma(1), \ldots, f_{\sigma(n)})$.

You can do this for cubes too, but it's very hard to write a map between the little disks to little cubes.

Let me finish this example. The composition is as follows. $\circ_i : D_k(n) \times D_k(m) \to D_k(n + m - 1)$ takes $(f_1, \ldots, f_n) \circ_i (g_1, \ldots, g_n) \mapsto (f_1, \ldots, f_{i-1}, f_i \circ g_1, \ldots f_i \circ g_m, f_{i+1}, \ldots, f_n)$. So here's an example [picture].

Let $(X, *)$ be a based topological space. Let $\Omega^k X = \{r : D^k \to X | r(\delta D^k) = *\}$. This is the $k$th iterated loop space. The way I've defined it is already nice. I can do an algebra map. We claim that $\Omega^k X$ is an algebra over $D_k$, so there is an operad map from $\mathscr{F} : D_k \to End_{\Omega^k X}$ which means that for all $n$ there are maps $\mathscr{F}_n : D_k(n) \to Hom(\Omega^k X)^n, \Omega^k X)$.

[Picture describing the map].

So for $t \in D^k$, given $r_1, \ldots, r_n$ from $D^k \to X$, $t$ maps to $r_j \circ f_j^{-1}(t)$ if $t \in f_j(D^k)$ for some $j$ and otherwise to the basepoint.