Deformation Theory and Operads

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Recall. What you need for an operad is a sequence of spaces $\mathcal{O}(n)$ for $n \geq 1$. You have a right Σ_n action on $\mathcal{O}(n)$, a $\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1)$ satisfying associativity and equivariance, and then a unit $\mathcal{O}(1)$ which satisfies a unit condition. The two examples were the endomorphism operad of a vector space and of a topological space. We also did Com with $Com(n) = k$ for $n \geq 1$ with the trivial action. I want to introduce a couple more operads today, associative and Lie, but before that I should talk about another example.

[What about the sign representation on Com?]

Tensoring with this $\widetilde{Com}$ gives an operad where A is an algebra over $\mathcal{O} \otimes \widetilde{Com}$ if ΠA is an algebra over $\mathcal{O}$. This is in a paper of Tamarkin's.

Let $\mathcal{O}$ be an operad in vector spaces such that $\mathcal{O}(n) = 0$ for all $n \geq 2$. So then $\mathcal{O}(1) = V$. This means that we have a right Σ_1 action. That's vacuous. Then we have an operation $\circ_1 : V \otimes V \to V$ satisfying associativity, along with a unit for $\circ_1$. In other words, we get an operad if and only if $(V, \mu, \mathbf{1})$ is an associative algebras. So associative algebras sit inside operads.

Let me make the definition of an algebra over an operad. First, let $\mathcal{O}, \mathcal{P}$ be two operads in $\mathscr{C}$ where this is either Top or $Vect$. Then a morphism of operads $\mathscr{F} : \mathcal{O} \to \mathcal{P}$ consists of a sequence of morphisms $\mathscr{F}_n : \mathcal{O}(n) \to \mathcal{P}(n)$ such that

1. $\mathscr{F}_n(f) \circ_i \mathscr{F}_n(g) = \mathscr{F}_{n+m-1}(f \circ_i g)$.
2. $\mathscr{F}_n(f \cdot \sigma) = \mathscr{F}_n(f) \cdot \sigma$
3. $\mathscr{F}_1(\mathbf{1}_{\mathcal{O}}) = \mathbf{1}_{\mathcal{P}}$

We call V an algebra over an operad $\mathcal{O}$ if we are given a map from $\mathcal{O}$ into the endomorphisms of V.

I guess the first interesting example is that a map $\mathcal{C}om \to End_V$ is really for each n a map $Com(n) \to Hom(V^{\otimes n}, V)$.

So $Com(n)$ is k so all you do is choose, well, start with $Com(1)$. This should go into $Hom(V, V)$ so it should be the identity, the unit maps to the unit. $Com(2)$ maps into $Hom(V^{\otimes 2}, V)$. This is the only information, as we'll see in a moment. Take the unit of the field and this maps to $\mu : V^{\otimes 2} \to V$. There is a σ_2 action here, and it is trivial. So the product μ is commutative.

All higher ones are generated by this; if you look at 1_3 in $Com(3)$, this is $1_2 \circ_1 1_2$, which is $\mu_3 : V^{\otimes 3} \to 3$. This means that $\mu_3(v_1, v_2, v_3) = \mathscr{F}_3(1_3)(v_1, v_2, v_3)$, which is

$$\mathscr{F}_3(1_2 \circ_1 1_2)(v_1, v_2, v_3) = \mathscr{F}_2(I_2) \circ_1 \mathscr{F}_2(I_2)(v_1, v_2, v_3) = \mu \circ_1 \mu(v_1, v_2, v_3) = \mu(\mu(v_1, v_2), v_3)$$

The outcome is that V is an algebra over Com if and only if (V, μ) is a commutative associative algebra.

Let's get to a more interesting example. The associative operad $Assoc(n)$ is the vector space spanned by non-commuting monomials in the variables $x_1, \ldots, x_n$, containing each x_i exactly once. This is the span of $\{x_{\sigma(1)}, \ldots, x_{\sigma(n)} | \sigma \in \Sigma_n\}$. So maybe let's do the lowest one. The lowest one is spanned by x_1. $Assoc(2)$ is spanned by $x_1 x_2$ and $x_2 x_1$; $Assoc(3)$ has six permutations. We need a unit, which will be x_1. The Σ_n action is that $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\rho = x_{\sigma\rho(1)}, \ldots, x_{\sigma\rho(n)}$. Check that this is an action, that $(\bar{x}_\sigma \rho)\rho' = (\bar{x}_\sigma)\rho\rho'$.

Composition of $(x_{\sigma_1}, \ldots, x_{\sigma_n}) \circ_i (x_{\rho_1}, \ldots, x_{\rho_m})$ is the substitution of $x_{\rho_1}, \ldots, x_{\rho_m}$ at the x_ith spot. So $(x_3 x_1 x_2 x_4) \circ_2 (x_3 x_2 x_1)$. This should be in $Assoc(6)$. It's $x_5 x_1 (x_4 x_3 x_2) x_6$. I put the parentheses to acknowledge the placment. They don't matter.

So what is an algebra over $Assoc$? It's $\mathscr{F}_n : Assoc(n) \to Hom(V^{\otimes n}, V)$. So the unit goes to the identity again, and then $x_1 x_2 \in Assoc(2)$ goes to $\mu \in Hom(V^{\otimes 2}, V)$. Then $x_2 x_1$ goes to $\tilde{\mu} \in Hom(V^{\otimes 2}, V)$. So $\mu(v_1, v_2) = \mathscr{F}_2(x_1 x_2) = \mathscr{F}_2(x_2 x_1 . \tau)(v_1, v_2) = \mathscr{F}_2(x_2, x_1).\tau(v_1, v_2) = \tilde{\mu}.\tau(v_, v_2) = \tilde{\mu}(v_2, v_1)$ so $\tilde{\mu} = \mu^{op}$. So the $\circ_i$ operation determine the higher operations in terms of the lower ones, and that the product is associative.

From the Σ_n action we see that

$$\mathscr{F}_n(x_{\sigma 1}, \ldots, x_{\sigma n})(v_1, \ldots, v_n) = \mathscr{F}_n(x_1, \ldots, x_n . \sigma)(v_1, \ldots, v_n) = \mathscr{F}_n(x_1, \ldots, x_n).\sigma(v_1, \ldots, v_n) = \mathscr{F}_n(x_1, \ldots, x_n)(v_\sigma$$

So this is the general scheme. If you have a structure and want to build the operad that governs it, we've done the associative and the commutative. We could do the commutative differently. The associative algebra was $x_1, \ldots, x_n$ and, well, let's see how I want to write this. Note that we could have also written $\mathcal{C}om(n)$ to be the span of all monomials with exactly one x_1 up to x_n living in the free symmetric algebra.

Let $Lie(n)$ be the span of all monomials with exactly one $x_1, \ldots, x_n$ in the free Lie algebra $L(x_1, \ldots, x_n)$. Recall that $(L, [\,,\,])$ is a Lie algebra if $[\,,\,] : L \otimes L \to L$ satisfies $[\ell_1, \ell_2] = -[\ell_2, \ell_1]$ and $[[\ell_1, \ell_2], \ell_3] + [[\ell_2, \ell_3], \ell_1] + [[\ell_3, \ell_1], \ell_2] = 0$ (Jacobi) You think of the free Lie algebra as anything you can build out of brackets. Any iterated brackets up to anticommutativity and Jacobi.

Recall that for a vector space V we have the free tensor algebra TV, the sum

$$\bigoplus_{n\geq 0} V^{\otimes n}$$

satisfying that it is free, namely for all vector space maps $V \to A$ there is a unique lift $TV \to A$ such that you have some commutative diagram.

$$\begin{tikzcd} & TV \arrow[d, "\tilde{f}"] \\ V \arrow[ur, "i"] \arrow[r, "f"'] & A \end{tikzcd}$$

This map just depends on the lowest component. You eventually see that $\tilde{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1)\cdots f(v_n)$. There is an induced Lie algebra structure on TV given by $[x, y] = x\otimes y - y\otimes x$. Then define LV to be the Lie subalgebra of $TV, [\,,\,]$ generated by V. This is explicitly the intersection of all Lie algebras inside TV containing V. This is LV and one of the goals is to show that this is the free Lie algebra generated by V. For each vector space morphism there is a unique lift to LV. It's unfortunately hard to compute or calculate with LV.

What I'd like to do is give a characterization of LV. This is not done very often. Note that $TV\otimes TV$ is an associative algebra: $(x\otimes y)\bullet(r\otimes s) = (x\cdot r)\otimes(y\cdot s)$ so using $\Delta : V \to TV\otimes TV$, which maps V to $V \otimes 1 + 1 \otimes V$, I just gave you a map of V into some algebra. Then using the universal property there is an induced algebra map $\tilde{\Delta} : TV \to TV \otimes TV$. How does this act? So $\tilde{\Delta}(v_1, \ldots, v_n)$ is determined by its lowest components: It's $\prod \Delta(v_i)$ So this is $(v_1 \otimes 1 + 1 \otimes v_1)\cdots(v_n \otimes 1 + 1 \otimes v_n)$, and multiplying this out you get, for each one, v_i in one or the other factor. This is called an unshuffle or a shuffle

$$\sum_{p+q=n} \sum_{(p,q)-\text{shuffles}\sigma} (x_{\sigma 1} \otimes \cdots \otimes x_{\sigma p}) \otimes \left(x_{\sigma(p+1)} \otimes \cdots \otimes x_{\sigma n}\right)$$

Where σ is a p, q shuffle if $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$.

Let me state, well, this is commutative. You can shuffle in the opposite order, so this is actually a commutative bialgebra. To continue, this is a Hopf algebra with antipode given by $S(v_1, \ldots, v_n)$ is $(v_n \otimes \cdots \otimes v_1)$

Now let $\gamma : TV \to LV$ be given by $\gamma(1_{TV}) = 0$ and $\gamma(v) = v$. Then

$$\gamma(v_1 \otimes \cdots \otimes v_n) = [v_1, [v_2, \cdots, [v_{n-1}, v_n]\cdots]]$$

Proposition 1 *For $x \in V^{\otimes n} \subset TV$ the following are equivalent:*

1. $x \in LV$
2. $\tilde{\Delta}(x) = x \otimes 1 + 1 \otimes x$ *(Friedrichs)*
3. $\gamma(x) = nx$ *(Dynkin-Specht-Weiner)*

I'm out of time.