Deformation Theory and Operads

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Let $mathcalA$ be the category of local Artin k-algebras with residue field k. Local means one maximal ideal. This should be commutative, associative, and unital. I have a map $k \to R$ and another map $R \to R/m$. I assume that R/m is isomorphic to k and that the composition of the reduction with the unit is an isomorphism. I don't need to say this if k is algebraically closed.

Artin by definition means that, it's some finiteness condition on ideals which translates into the following: Such rings R are isomorphic to $k \oplus m_R$ where m_R is finite dimensional and nilpotent. In fact, there's a structure theorem giving each R as a polynomial ring in a finite number of variables modulo an ideal which is nilpotent, where some power of t is zero. As an example, $k[t_1, t_2, t_3]/\langle t_1^4 - t_2^2 t^3, t_2^1 0, t_3^2 \rangle$. The morphisms are morphisms of k-algebras.

Let $\hat{mathcalA}$ be the category of complete local Noetherian k-algebras, let me say it this way, of Noetherian projective limits of local Artin algebras. These are local rings with a maximal ideal m such that R/m is isomorphic to k and R/m^n is in A for each n. In fact R is the limit of the diagrams $\cdots \to R/m^3 \to R/m^2 \to R/m = k$. Every chain of ideals stabilizes, $I_1 \subset I_2 \subset \cdots$. Every ascending chain of ideals is finite. Artin means every descending chain is finite, and that is stronger and implies Noetherian.

Example: $k[t]/t^5$ is Artin. The ideals are spanned by t, t^2, and so on. There are only four nontrivial ideals, and so there is just one chain, which is finite. Now, $k[[t]]$ is Noetherian. You have the ideals generated by t^n. But $k[t_1, t_2, \ldots]/\langle t_i^2 \rangle$ is not Noetherian and you don't want to consider that. This diagram does not relate well to topology.

Definition 1 *A functor of Artin rings is a covariant functor $mathcalA \to Sets$. An element $\gamma \in F(A)$ is called a deformation of $\gamma_0 \in F(A/m)$ where γ_0 is the image of γ under the map $F(A) \mapsto F(A/m)$ where this is the induced map.*

We define the tangent space to F to be $t_F = F(k[\epsilon])$ where $k[\epsilon]$ means $k[\epsilon]/\epsilon^2$.

Let's look at some examples. Let $(V, \bullet)$ be an associative k-algebra. Define $F(A)$ to be $F(A) =$ Deformations of $(V, \bullet)$ over A which is associative products from $(V \otimes m_A)^{\otimes_A 2} \to V \otimes m_A$. Recall if (A, ϵ) is an augmented ring, where $\epsilon : A \to k$ then a deformation of $(V, \bullet)$

over (A, ϵ) is an associative product μ on $V \otimes A$ such that the map $V \otimes A \to V \otimes k \to V$ is a map of algebras.

Given an Artin local ring A we have an augmentation $\epsilon : A \to A/m \cong k$. So such a deformation is an associative product μ such that $V \otimes A \to V$ is a map of algebras. Since $A \cong k \oplus m$ this is $V \oplus V \otimes m \to V$. The part in the V cannot change, and the map kills the m side.

How do I define f on morphisms? Given a deformation over A and a homomorphism over B I need to define a homomorphism over B. We can express $\mu(v, w) = vw + b(v, w)$ where $b(v, w) \in V \otimes m_A$. So I can define a new product on $V \otimes B$ by $\mu_\sigma(v, w) = vw + \sigma(b(v, w))$.

The next example would have this be deformations modulo equivalence and I'd also get a functor. But here's the example that I like. Fix an R in $\hat{A}$ and define $h : mathcalA \to Sets$ by $h_R(A) = hom(R, A)$. This defines a functor, and given a morphism $\sigma : A \to B$ then $h_R(\sigma)$ is defined as $hom_{A^\sim} : hom(R, A) \to hom(R, B)$ which takes f to σf.

Definition 2 *A functor F of Artin rings is rpepresntable by $R \in \hat{A}$ if and only if F is naturally equivalent to h_R for $R \in \hat{A}$. Remark: suppose $F = h_R$ for some $R \in \hat{A}$. Then $F(A) \cong hom(R, A)$ for all $A \in \hat{A}$. However, any $F : mathcalA \to Set$ may be extended to $\hat{F} : \hat{A} \to Set$ by $\hat{F}(S) = lim\ F(S/m^n)$. This is a sequence. An element in $F(S)$ is a sequence of elements in $F(S/m^n)$*

You end up replacing the set of deformations with a set of ring homomorphisms.

The functors extend naturally to limits of Artin rings, so in particular, $F(R) \cong hom(R, R)$.

There exists a $\Gamma \in F(R)$ so that Γ corresponds to the identity in $hom(R, R)$.

There's a particular deformation which satisfies a universal property. Notice that Γ satsifies a universal property: Given any $A \in mathcalA$ and $\gamma \in F(A)$, there exists a unique ring homomorphism $R \to A$ such that $F(\sigma)(\Gamma) = \gamma$. This is the end of my remark and is a good stopping place. You come up with this when you chase the diagrams around.