Deformation Theory and Operads

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How does the Hochschild cohomology of an algebra A compare to the Hochschild cohomology of a deformation of A? Let A be an algebra over k and let $\mu : A[[t]] \otimes_{k_t} A[[t]] \to A[[t]]$ be a formal one parameter deformation of A.

Theorem 1 *The rank over k_t of $C^i(A_t, A_t)$ is less than or equal to the rank over k of $C^i(A, A)$.*

Let's prove this by discussion. Let $\Psi \in C^1(A_t, A_t) = Hom_{k_t}(A_t^{\otimes i+1}, A_t)$. Then Ψ is determined by $\Psi(a, b) = \psi_0(a, b) + \psi_1(a, b)t + \cdots$

Okay, so Ψ looks like that. What about Φ in $C^0(A_t, A_t)$? That's determined by its value on a as $\phi_0(a) + \phi_1(a)t + \cdots$ for $a \in A$. Talking about automorphisms, we assumed on Monday that the first term was the identity. Now we don't need to assume that, it doesn't need to be invertible for instance.

What is $\partial_t \Phi$? This takes two arguments, so

$$\partial_t \Phi(a, b) = \mu(\Phi a, b) - \Phi(\mu(a, b)) + \mu(a, \Phi(b))$$

Both μ and Φ are expandable in powers of t. Doing this we see that, well, notice that the right hand side is equal to

$$\sum_{i+j=n} f_i(\phi_j(a), b) - \phi_j(f_i(a, b)) + f_i(a, \phi_j(b))$$

In particular you get for $n = 0$ that $\partial \phi_0 = \phi_0(a)b - \phi_0(ab) = a\phi_0(b)$. So there's a chain map from $C^i(A_t, A_t)$ to $C^i(A, A)$.

First we have this chain map by setting $t = 0$, so $\pi\delta_t = \delta\pi$. This is especially clear if you view boundary as bracketing with multiplication, because then you can write the output of bracketing in powers of t and you get a sequence of operators on the left hand side. Really I want to just take a couple more minutes.

How do you prove the theorem? First, let Z_t and Z be cocycles in $C^i(A_t, A_t)$ and $C(A, A)$ and B_t and B the coboundaries. Observe that $rk_{k_t} Z_t^i \leq rk_k Z^i$. By construction, the constant term in any cycle is a cycle. This may not be onto. There may be cycles which cannot be extended by adding t terms. If I have an element $\psi_0 + \psi_1 t + \cdots$ which is closed under δ, that's an infinite number of relations. I definitely get a cycle. On the other hand, $rk_{k_t} B_t^i \geq rk_k B^i$ and I'm just going to end here. In other words, you can have things that are boundaries in the deformed complex which go to zero. There could be k_t torsion here. If you write down the equation it will be clear. I don't want to rush because it will seem hard, so I will just stop.