Deformation Theory and Operads

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Recall that D_k is the little k-disks operad. It acts on $\Omega^k X$ for any topological space X.

Theorem 1 *Weak Recognition principle (May 1972, Bordmann Vogt 1973)*
If X is a path connected topological space such that X is an algebra over D_k, then there exists a space Y so that $X \cong \Omega^k Y$, they are homotopy equivalent

Remark.

1. There exists an operad WD_k such that X is an algebra over WD_k if and only if X is homotopy equivalent to a k-fold loop space $\Omega^k Y$.

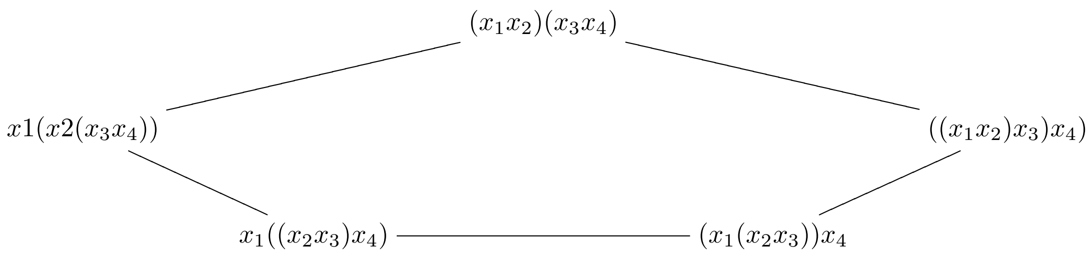
   W is basically the free operad construction on D_k in the topological category.

2. X is called an infinite loop space if there exists a sequence of spaces $X_1, X_2, \ldots$ such that $X \cong \Omega X_1$, $X_1 \cong X_2$, and so on. There is a strong recognition principle with the following. A topological operad is called an E_∞ operad if each $O(n)$ is Σ_n-free and each $O(n)$ is contractible. Now X is an algebra over an E_∞ operad so that X is an infinite loop space. For example, take D_∞ to be the direct limit of D_k, where D_k sits inside D_{k+1}. This limit is an E_∞ operad and so you could, this answers your question.

Let me give small indications for the weak recognition principle. This was first done when $k = 1$ by Stasheff. Assume that X is a D_1 space. This is the little interval operad. We want to show that X is homotopy equivalent to ΩY. This is in H-spaces in Lecture Notes in Mathematics in 1963.

This is in two steps. The first step we show that there are polytopes K_n for $n \geq 2$ and maps $K_n \to D_1(n)$. This is the first example of this W. The K_n assemble to an operad K where $K(n) = K_n \times \Sigma_n$ and K is essentially equal to WD_1. Spaces over this K are called A_∞ spaces. Whenever you have an algebra over this K you can define the following, let Y be the union for $n \geq 0$ of $K_{n+2} \times X^n / \sim$, so you just, take this realization, put these together, and you divide out by equivalence and glue together and then one can check that X is homotopy equivalent to ΩY. Let me go back to step 1. The second step uses the notion of quasifibration and he builds this up from, starting up to m. This uses a lot of topology machinery. The nice thing that is really neat are these associahedra. Define K_n as the convex polytope with one vertex for each way of associating n ordered variables $x_1, \dots, x_n$. For example, with six variables, I could associate with $x_1((x_2x_3)(x_4(x_5x_6)))$. This is in correspondence with trivalent ordered trees.

What are the low dimensional examples? K_2 is a point. It's just, (x_1x_2). Now K_3 is, well, you have two points, $(x_1x_2)x_3$ and also $x_1(x_2x_3)$ and you have to connect them and you get an edge. K_4 there are five ways of associating



Better than the convex polytope with this as its hull is to take $\sqcup K_r \times K_{n-r+1}/\sim$ and take the cone on this union L_n, I get something of dimension $n-2$. One can check that L_n has the homotopy type of the sphere, which is highly nontrivial. This was shown by Adams. I think I will stop here.

[Discussion of $\sim$]

[Picture of K_5]

Something I want to say is what happens when you take the homology of little disks. Let $F : Top \to Vect$ be a functor respecting $(\times, \sqcup) \to (\otimes, \oplus)$. If you have an operad in $\mathscr{C}$ then define $F(O)$ be the operad in C' given by $F(O) = \{F(O(n))\}$ for $n \geq 1$. You just drag everything through, so that $\circ_i = F(\circ_i)$ and Σ_n action is just $F(\sigma_n)$, and the unit is $F(1_O)$. I should have said that $*$ should map to k, it's a symmetric monoidal functor. If it respects all the structure, then obviously all the relations are satisfied. Then the proposition is

Proposition 1 *Let $k \geq 2$ and let $P_k = H_*(D_k)$. Then a graded vector space V is an algebra over P_k if and only if V is a $k-1$-algebra, where that is to be defined:*

Definition 1 *To say that V is a $(k-1)$-algebra means three things.*

1. *V is an associative and graded commutative algebra with product $\bullet$*
2. *$V[k-1]$ is a graded Lie algebra with bracket $\{\ ,\ \}$ (the degree of the bracket is $1-k$)*
3. *There is a Liebnitz relation with the bracket $\{u, v \bullet w\} = \{u, v\} \bullet w + (-1)^{(|u|+k-1)|v|} v \bullet \{u, w\}$*

What are these two operations? Identify two homology elements. Any old point as an element in $D_k(2)$, this is the product. For the bracket, it's a $k-1$-sphere of spheres moving around a sphere.

There was one more thing I wanted to say, to clarify how you see an operad as a monoid in a monoidal category. I'm not sure I should do that. Let me do this, it's fairly fast. It's an alternative definition of operads, of what an operad is.

Denote by $\Sigma - Vect$ the category with objects $A = \{A(n)\}_{n \geq 1}$ where each $A(n)$ has a Σ_n-right action and morphisms are maps $f = \{f(n)\}_{n \geq 1}$ where $f_n : A(n) \to B(n)$ is an equivariant vector space map. There is a monoidal structure called $\boxtimes$ on $\Sigma - Vect$ such that an operad on vector spaces is the same as a monoid in this monoidal category.

The idea is if you have, for A and B as having n inputs, take $(A \boxtimes B)(n)$ to be trees with trees on them. On the bottom you put A and on the top B. You take the sum over all trees with n inputs, split into two levels with Bs on top. More precisely, you have to go through the following steps. First of all, you extend $A(n)$ to any finite set X as an input, so given X and A in $\Sigma - Vect$ let $A(X)$ be the sum over bijections $X \to n$ of $A(n)$ modulo some Σ_n action. Then in the second step, you can define for n and m at least 1 and an object A, let $A(n, m)$ be the sum over surjective maps $n \to m$ of $A(f^{-1}(1)) \otimes \cdots A(f^{-1}(m))$. Finally, for the third step, you let $A \boxtimes B$ be the sum of $A(m) \otimes_{\Sigma_n} B(n, m)$.

The fact is that this $\Sigma - Vect, \boxtimes$ is a monoidal category with unit $U = U(1) = k$. A monoid in $\Sigma - Vect$ is an object in $\Sigma - Vect$, an object A together with morphisms $\mu : A \boxtimes A \to A$ and $\eta : U \to A$. This should have the usual commutative diagrams for associativity of μ and the unit properties of η. Such a monoid is exactly an operad. Wednesday there is a seminar, Loday, so no class.