

# Deformation Theory and Operads

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Remember that we were talking about functors  $F : \mathcal{A} \rightarrow \text{Set}$  where  $\mathcal{A}$  was the category of Artin local rings. We had a set of conditions:

H0  $F(k)$  is a point

H2'  $\alpha : F(A \times_C B) \rightarrow F(A) \times_{F(C)} F(B)$  is a bijection for  $A = B = \mathbf{k}[\epsilon], C = \mathbf{k}$

H2  $\alpha$  is a bijection when  $A = \mathbf{k}[\epsilon], C = \mathbf{k}$

H3  $F(k[\epsilon])$  which is the definition of the tangent space  $T_F$  is finite dimensional.

H4  $\alpha$  is a bijection for  $A \rightarrow C$  small with  $B = \mathbf{k}$ , so that  $F(A') \rightarrow F(A)$  is a bijection

The first two of these say that  $T_F$  is a vector space; addition comes from  $\mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon] \rightarrow \mathbf{k}[\epsilon]$

[Discussion broken by H4]

## 1 Differential Graded Lie algebras

A differential graded Lie algebra, or dgLa, is a triple  $(V, d, [\cdot, \cdot])$  here  $V$  is a  $\mathbb{Z}$ -graded vector space,  $d : V^i \rightarrow V^{i+1}$  is a square zero degree one operator,  $[\cdot, \cdot]$  is a degree zero bilinear operator on  $V$  which is graded skew symmetric and satisfies graded Jacobi:

$$[x, y] + (-1)^{|x||y|}[y, x] = 0$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$$

The compatibility between differential and bracket is

$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$$

I think of this as saying that bracketing with  $x$  is a derivation of degree  $|x|$  of the bracket, and  $d$  is a derivation of the bracket.

If  $x$  is even, then  $[x, x] = 0$ . If  $x$  is odd, then  $[x, [x, x]] = 0$ . Also,  $[x, [x, y]] = [[x, x], y] + (-1)^{|x||x|}[x, [x, y]]$  so that  $2[x, [x, y]] = [[x, x], y]$ .

If  $C = (W, d, \bullet)$  is a graded commutative associative algebra and  $L = (V, d, [, \cdot])$  is a dgLa, then  $L \otimes C$  which is defined as  $(V \otimes W, d \otimes 1 + 1 \otimes d, [, \cdot])$  is a dgLa.

This could have been discussed in terms of chain complexes. The commutative operad has just one operad in each arity; tensoring with the commutative operad is trivial, and so tensoring with a commutative algebra preserves the type of algebra you started with.

The formula for the bracket in  $L \otimes C$  says that  $[v \otimes a, w \otimes c] = (-1)^{|a||w|}[v, w] \otimes ab$ .

Now I want to define a functor of Artin rings, fixing a dgLa. One may deform  $d$  of a dgLa in the direction of an inner derivation. You want the new operator  $d_a := d + [a, -]$  to be a square zero degree one derivation of the bracket. It will always be a derivation; for the grading to be correct you want  $a$  to be in degree one. If  $d_a + \frac{1}{2}[a, a] = 0$  and  $a \in V^1$  then  $d_a$  squares to zero.

$$d_a^2(x) = d^2x + d[a, x] + [a, dx] + [a, [dx]] + [a, [a, x]] = [da, x] - [a, dx] + [a, dx] + [a, [a, x]]$$

If  $a$  is odd then this means  $[da + \frac{1}{2}[a, a], x] = 0$ . This leads to the equation for an element of degree one satisfying  $da + \frac{1}{2}[a, a] = 0$ , which is called the Maurer Cartan equation or the classical master equation, or even the deformation equation.

An observation: given a graded Lie algebra and an element  $a \in V^1$  satisfying the equation  $[a, a] = 0$ , one has a differential graded Lie algebra  $(V, d, [, \cdot])$  where  $dx := [a, x]$

Let's look the other way around. Given a dgLa  $L$  define a new dgLa by  $L' = (V \oplus kd, [, \cdot]_{new})$  where  $[v + \alpha d, w + \beta d] = [v, w] + \alpha dw + (-1)^{|v|}\beta dv$  which lives in  $V \subset V \oplus kd$ . Then  $[d, d]_{new} = 0$  so we have a differential on  $L'$  defined by  $d' = [d, -]$ . One thing I didn't check was the Jacobi identity here. Let me go on. Now  $d'(v + \alpha d) := [d, v + \alpha d]_{new} = dv$ . I should have said that the degree of my new piece is 1. So now I get a new differential graded Lie algebra of the form  $L' = (V \oplus kd, d' = [d, -], [, \cdot]_{new})$ . The idea is that by extending the Lie algebra, I can think of the differential as being bracketing with a degree one element. So it's an extension of  $L$ , which is a subalgebra.

In this context, we have a map  $L \rightarrow L'$ , and I'll give a new map,  $a \mapsto a + d$ . Notice that, I would like this to be a map of the degree one part of  $L$  into the degree one part of  $L'$ ; well, let me say it like this: if  $a \in V^1$ , then  $[d + a, d + a]_{new} = 0$  in  $L'$  if and only if  $da + \frac{1}{2}[a, a] = 0$  in  $L$ .

Recall that in the Hochschild complex, well, if we have  $V$  we have  $HC^\bullet(V, V) := \bigoplus Hom(V^{\otimes n}, V)$ . There is a bracket on this complex, making it into a graded Lie algebra. If  $\bullet : V^{\otimes 2} \rightarrow V$  is a degree one element, and  $[\bullet, \bullet] = 0$ , then  $\bullet$  is an associative product. That was if and only if. Then we obtain a differential graded Lie algebra  $(HC^\bullet(V, V), d = [\bullet, -], [, \cdot])$ . I sort of think of the fundamental equation being  $[a, a] = 0$ .

Now, let's define, well, let  $L$  be a differential graded Lie algebra, and define a functor of Artin rings  $MC_L : \mathcal{A} \rightarrow Set$  by  $MC_L(A) = \{\gamma \in (L \otimes m_A)^1 | d\gamma + \frac{1}{2}[\gamma, \gamma] = 0\}$ , which are equal to deformations of the differential  $d$  in the direction  $\gamma$ , including parameters from  $A$ .

Given an associative algebra  $(B, \bullet)$ , define a  $F_B : \mathcal{A} \rightarrow Sets$  by  $F_B(A)$  are deformations of  $(B, \bullet)$  over  $A$ . Let  $L = (HC(B, B), d = [\bullet, ], [ , ])$ . Then  $MC_L \cong F_B$  as functors of Artin rings. I believe that this statement doesn't need an argument.

Let's look at, let's illustrate the bijection. Start with  $\gamma \in MC_L(A) \subset HC(B, B) \otimes m_A$ . So  $\gamma : B \otimes B \rightarrow B$ . So  $\gamma \in Hom(B \otimes B, B) \otimes m_A$ , which is equal to, I claim,  $Hom(B \otimes B, B \otimes m_A)$ . Then  $\gamma(b_1, b_2) \in B \otimes m_A$ . Define a product  $\bullet_\gamma : (B \otimes A)^{\otimes 2} \rightarrow B \otimes A$  by  $\bullet_\gamma(b_1, b_2) = b_1 \bullet b_2 + \gamma(b_1, b_2)$  where these pieces are in  $B \otimes \mathbf{k}$  and  $B \otimes m_A$  so their sum is in  $B \otimes A$ .

Maurer-Cartan equation for  $\gamma$  implies associativity for  $\bullet_\gamma$  and the new product when you map down to  $A$  gives you the old product. I claim that's a bijection. Remember that a deformation of  $B$  over an augmented ring. Modding out by the maximal ideal, you should get the old thing. So it's exactly something of this form. The associativity gets you Maurer Cartan for  $\gamma$ .