Deformation Theory and Operads

Gabriel C. Drummond-Cole

July 9, 2012

[Missed a lot] The basic setup is a dgLa with a group action. You can look at derivations and the part you care about is the deformations of a given derivation by the inner derivations of degree one. So $d + ad\ \lambda$. Now L_0 acts on this space, by $d + ad\ \alpha \mapsto^{\lambda} d + ad\ ([\lambda, \alpha] - d\lambda)$. I computed this.

$[ad\lambda, d{+}ad\ \alpha] = [ad\ \lambda, d]\beta{+}[ad\ \lambda, ad\ \alpha]\beta = [\lambda, d\beta]{-}(-1)^{|ad\ \lambda||d|}d[\lambda, \beta]{+}[\lambda, [\alpha, \beta]]{-}\alpha, [\lambda, \beta]] = [\lambda, d\beta]{-}[d\lambda, \beta]{-}[\lambda, d$

The result is that this is as stated in the claim.

If d is an element in the Lie algebra I don't really have to say anything else. So you have a Lie algebra acting on a tangent space, so the associated Lie group should act on the space. Let me start the other way around. If a Lie group G acts on a manifold M then the Lie algebra of G acts on the tangent space of M.

If I take $ad\ L^0$ acts on L^i, namely $\lambda : L^i \to L^i$ by $\alpha \mapsto [\lambda, \alpha]$. This just gives me an action on this vector space and I can exponentiate these morphisms and we have the exponential of L^0. You might ask if that's defined. I can just take that as the definition of, it's thase elements of $GL(L^i) : \lambda \in L^0$.

By definition $e^{ad\ \lambda}(\alpha) = \alpha + [\lambda, \alpha] + \frac{1}{2!}[\lambda, [\lambda, = alpha]]$

Let's do a calculation. $e^{ad\ \lambda}(\alpha + \beta) = e^{ad\lambda}\alpha + e^{ad\lambda}\beta = e^{ad\ \lambda}\alpha + \beta + [\lambda, \beta] + \frac{1}{2!}[\lambda, [\lambda, \beta]] + \cdots = e^{ad\ \lambda}\alpha + \beta + [\lambda, \beta] + \frac{1}{2!}ad\ \lambda([\lambda, \beta]) + \frac{1}{3}ad\ \lambda^2([\lambda, \beta]) = \beta + e^{ad\ \lambda}\alpha + \left(\frac{e^{ad\ \lambda} - 1}{ad\ \lambda}\right)[\lambda, \beta] = \beta + e^{ad\ \lambda}\alpha + \left(\frac{1 - e^{ad\ \lambda}}{ad\ \lambda})[\beta, \lambda]\right)$.

In conclusion, let L be a dgLa, and then form $\tilde{L} = L \oplus kd$. Last time I defined the bracket with this, to make this a new Lie algebra. Then if I take $e^{ad\ \lambda}$ applied to $(d + \alpha)$, this will be a computation in $\tilde{L}$. So $e^{ad\ \lambda}(d + ad\ \alpha)$. Then the computation I did says that this is $d + e^{ad\ \lambda}\alpha + \left(\frac{1 - e^{ad\ \lambda}}{ad\lambda}\right)[d, \lambda] = d + e^{ad\ \lambda}\alpha + \left(\frac{1 - e^{ad\ \lambda}}{ad\lambda}\right)$. This shows that the action of $exp(L^0)$ on A is given by $d + ad\ \lambda \mapsto d + \left(e^{ad\ \lambda} + \left(\frac{1 - e^{ad\ \lambda}}{ad\ \lambda}\right)(d\lambda)\right)$. This is a group version of this map $\alpha \mapsto [\lambda, \alpha] - d\lambda$.

This is a special case related to the Campbell Baker Hausdorff formula.

Okay. Now I wanted to do a computation showing that, this is something I can show for any such α. Then I can, and so, it will give me a way, this is what I have so far. The exponential of L^0 acts on L^1, hence on $ad\ L^1$ and so on $A \subset Der(L)^1$. The conclusion is that this preserves solutions to the Maurer Cartan quation. Let me put another claim down to say this.

If $Q(\alpha) = d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$ then $Q(e^\lambda \alpha) = d(e^\lambda \alpha) + \frac{1}{2}[e^\lambda \alpha, e^\lambda \alpha] = 0$.

This gives you a definition. YOu can define an equivalence relation on the set of solutions which I am going to call MC. This says $\alpha \sim \alpha'$ for some $\lambda \in L^0$. Now define a functor from $\mathcal{A}$ to sets by $Def^1_L \otimes m_A : Q\alpha = 0$ modulo equivalence.

Then there's a computation. This says that T_{Def_L} which is $Def_L(\mathbf{k}[\epsilon])$ is the same as $H^1 L$. So deformations correspond to closed things and equivalences correspond to exact ones.

Definition 1 *Make an equivalence relation on $MC_L(A)$ if and only if there exists an ω in $(L \otimes m_A)[t, dt]$. So $d\omega + \frac{1}{2}[\omega, \omega] = 0$ and $\omega(0,0) = \alpha_0$ while $\omega(1,0) = \alpha_1$. Here $d\omega$ should be interpreted as the sum of the two d, the one on t and the one in L. You can view α as a differential coalgebra map from $(m^*_A, 0) \to (SV, D)$.*

Then $\alpha_0 \sim \alpha_1$ if and only if they are equivalent using this formula. This is important because the group action version of equivalence is pretty special to Lie algebras.

I actually did these claims, proved them, and then I didn't work out a full proof for one of them. Anyway, I don't want to do that, I'd rather stop and make some closing remarks.

We talked about A_∞ and Hochschild. We here broached L_∞ and that has Chevelley-Eilenberg, and then C_∞ and Harrison. I talked about L_∞ in terms of controlling deformations of other problems. You could try to deform L_∞ algebras and you'd get the Chevelley-Eilenberg, which is itself L_∞ You have Tor and Ext type interpretations of these things. This leads directly into the important idea of Koszul duality. This is a kind of statement you can make at the level of operads and even more generally. Associative is self-dual, and commutative and Lie are.

Another important idea is the operad of little disks, and there's also framed little disks. This leads to BV algebras. Little disks lead directly to Kontsevich's formality theorem. It says a certain kind of Hochschild complex is quasiisomorphic to its homology. All of the deformations of that associative algebra are unobstructed. That deformation functor is representable. There is a whole body of work from the 70s forward about this, called deformation quantization of Poisson manifolds. You want to pass from functions on a symplectic manifold to operators on a Hilbert space. They commuted before and now they won't commute. This is in terms of deforming a commutative associative product into a noncommutative associative product, using deformations in one variable. Kontsevich showed that if your algebras are functions on a manifold and if your [unintelligible]are [unintelligible]on the [unintelligible], it's always possible. This result had many partial results, but he gave the final result. The best proof of it is Tamarkin's proof, which makes an argument about little disks. There's a statement about little disks, which implies this statement at the level of operads.

This also gets into, I had some fantasy that we could do these other topics. I wasn't sure if we'd get into loop spaces and string topology. Of course, we touched on these topics a little bit when Thomas did the Hochschild complex, but there's a lot more structure involved in the Hochschild complex of the cochains on a manifold. It's not clear that all of the structure there has even been discovered. There are definitely research projects there. Also understanding string topology in terms of deformations. There should be an implicit geometric construction of the Hochschild homology. This is all even closely related to little disks. A lot of string topology, there are these ideas, quantum which can have a line on itself, which is more a research topic. Another topic is properads, which are many inputs and one output. Properads are many to many. There's a discussion about all of the other things. I'll put two more topics down. Complex manifolds, historically, were the first examples of deformation problems. What Gerstenhaber did was try to mimic what was done for complex manifolds. I didn't do any of this here. It immediately launches into topics of extreme relevance now. You have the B-model. You make one change, studying functors of graded Artin rings and then extended complex structures; there's, you know, people writing PhD theses on these. Geometrically they're mysterious. Another nice connection is to deformations of group representations. A couple of people in the 80s deformed the fundamental group of a K ahler manifold. They did all of these things we've talked about and this gave the Goldman bracket which is very closely connected to string topology.

That's sort of the end of the class. Do you have any closing remarks, Thomas? We have some possible research projects, Thomas. I said at the beginning that deformations are involved in some of the most celebrated proof. Ten years ago, the Shimura-[unintelligible]-[unintelligible]theorem, they used deformations of Galois representations. Then you have Kontsevich's result, little disks, this was, well, this was a Fields medal result in the 90s. Perelman used deformations of Riemannian metrics on manifolds. I think that the Ricci flow, he was thinking of it in terms of the string field action functional from physics.