Deformation Theory and Operads

Gabriel C. Drummond-Cole

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Recall that for a dgLa $V, d, [\,,\,]$ we have a functor $MC_L : \mathcal{A} \to Sets$ defined by $MC_L(A) = \{\gamma \in (L \otimes m_A)^1 : d\gamma + \frac{1}{2}[\gamma, \gamma] = 0\}$.

There is also a construction, given an associative algebra $(B, \bullet)$ one has the dgLa $HC^*(B, B), d = [\bullet, \,], [\,,\,]$, where the bracket exists for any old vector space.

The functor $F_B : \mathcal{A} \to Sets$ given by $F_B(A) = \{\text{deformations of } B, \bullet \text{ over } A\}$ is natrually equivalent to MC_L.

Reminder, fiven a deformation of $B, \bullet$ over A, $\bullet_A : (B \otimes A) \otimes (B \otimes A) \to (B \otimes A)$ it is determined by a map $B \otimes B \to B \otimes A$. The condition that $(B \otimes A, \bullet_A) \to (B, \bullet)$ is an algebra map implies that $\bullet_A : B \otimes B \to B \otimes A = B \otimes (k \oplus m_A) = B \oplus B \otimes m_A$, so this has the form $\bullet_A = \bullet + \alpha$ for some $\alpha : B \otimes B \to B \otimes m_A$.

Okay, then notice that α, view it in $Hom(B \otimes B, B) \otimes m_A \in HC(B, B) \otimes m_A$. Then we want to check that associativity of $\bullet_A$ implies that $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. This defines the map $F_B(A) \to MC_L(A)$ namely that $\bullet_A \to \alpha$, which is a bijection.

There's only one class left, so I'm going to have difficulty saying everything, but here's an important statement: given any "deformation functor" $F : \mathcal{A} \to Sets$ there exists a dgLa L so that $F \cong MC_L/\sim$. This is a statement, not a theorem, because it's not precise. I didn't give the axioms for it. I'd also have to talk about equivalence. One typically isn't interested in deformations over a ring, but the equivalence classes of these things.

What I want to take up next are equivalences on the Maurer Cartan solutions.

You have, generally, an object $\mathscr{O}$ and then a notion of deformations of $\mathscr{O}$ over parameter rings, and this leads to a functor $\mathcal{A} \to Sets$, and you want this to factor through Lie algebras so that you get L to make this functor the same as $MC_L/\sim$. What about extending? There are many simple ways of extending MC_L algebraically. For example, you can extend it to differential graded rings. A very popular question is that, if I start with some manifold, and I know that I get a dgLa, what if I extend the dgLa in some way? What do those mean geometrically? What are the new solutions. Right now there's an industry of people following Hitchens, but there are like 20 or 30 papers on extended complex structures or so on. In the algebraic example of associative algebras, if I extend the Lie algebra, they are deformations of the associative algebra as an A_∞ algebra.

Theorem 1 *(Deligne, Stasheff, Schlessinger, Kontsevich) $MC_L/\sim$ is naturally equivalent to $MC_{L'}/\sim$ if and only if the dgLa's L and L' are quasiisomorphic (as L_∞ algebras)*

A dgLa map $\sigma : L \to L'$ is a quasiisomorphism if and only if $\sigma_* : HL \to HL'$ is an isomorphism.

Now L and L' are called quasiisomorphic if they are equivalent under the equivalence relation generated by quasiisomorphisms.

There is an idea of L_∞ algebras. You make a definition of that, and then a definition of their morphisms. There are more L_∞ morphisms than dgLa maps. Two dgLa's are quasiisomorphic if and only if there is an L_∞ map between them inducing an isomorphism on homology.

I believe that if you replace L with A_∞ algebras, that corresponds to functors of noncommutative Artin rings. You lose information because a lot of things are not commutative.

[What about involutive biLie?]

I wish I knew the answer, I have a suspicion, I'm not saying it aloud.

Maybe I'll say one or two words about L_∞ algebras since they appeared and maybe it'll help a little. The questions you just asked, Gabriel, do you remember when you said a sequence of statements I made were a bankrupt point of view?

[That sounds rude.]

Well, these questions lead to a bankrupt point of view. Recall, given a vector space V there is the construction $SV = \bigoplus S^n V$, which has a co-commutative co-associative coproduct and in fact is the cofree cocommutative coalgebra on V. You take $v_1 \cdots v_n$ and you break that up in all distinct ways, $1 \otimes v_1 \cdots v_n + v_1 \otimes (v_2 \cdots v_n) + v_2 \otimes (v_1 \cdot \hat{v_2} \cdots v_n) \ldots$, with signs.

Then a coderivation $D : SV \to SV$ is determined by a sequence of maps $D_i : S^i V \to V$ for $i = 0, 1, \ldots$ An L_∞ algebra is a pair (V, D) where V is a graded vector space and D is a coderivation from $SV[1] \to SV[1]$ satisfying $D^2 = 0$ with degree $+1$ and no d_0 term.

I think that, here's whta I would like to say. L_∞ algebras are "free" differential coalgebras. I didn't say what a morphism is, but it's a map from SV to SW which is a cocommutative coalgebra respecting the differential.

Now what if I take an Artin algebra A with maximal ideal m_A? Then m_A is a finite dimensional commutative algebra. I can dualize it and get m_A^* which is a finite dimensional cocommutative coalgebra.

Here's an interpretation of the Maurer Cartan equation. Given L a dgLa, fashion the differential d and bracket $[\,,\,]$ into a codifferential $D : S(V[1]) \to S(V[1])$. This means you look at the differential as the map $V \to V$ and the bracket $S^2V[1] \to V$, add, and lift this to a coderivation.

You can define a functor $MC_L : \mathcal{A} \to Sets$ by $MC_L(A) = \{\gamma : m_A^* \to (SV[1], D)\}$: γ is a differential coalgebra map}

This is naturally equivalent to the functor I defined before. Let me write that down. $\gamma \in V^1 \otimes m_A \cong Hom^0(m_A^*, V[1])$, linear vector space maps, and every linear map, any linear map from a coalgebra into a vector space, using the fact that the free cocommutative coalgebra I get that every linear map corresponds exactly to a coalgebra homomorphism m_A^* into the free coalgebra, $Hom(m_A^*, S(V[1]))$. Remember, there is a condition on γ. This should satisfy the Maurer Cartan equation, which says that this map satsifies the differential, it's a differential coalgebra map.

Okay, there's a statement, I'm really giving a completely extemporaneous lecture. Now let me say something about the word extension. Note that it is easy to extend this to a functor $\tilde{MC}_L : \tilde{A} \to Sets$ where $\tilde{A}$ is differential graded Artin rings. Namely, given a differential graded Artin algebra (A, d_A), (m_A, d_A) is a finite dimensional commutative associative differential graded algebra and m_A^*, d_A^* is a differential graded cocommutative coalgebra, then $MC_L(A, d_A)$ is the set $\{\gamma : (m_A^*, d_A^*) \to (SV[1], D)\}$. What will it look like to represent this functor?

Let me suppose that you have a representing couple $(\Gamma, (R, D))$ which represents $\tilde{MC}_L$. This means you have a sequence of rings (R_i, D_i) and Γ_i is $MC_L(R_i, d_i)$. These are nice polynomials and Maurer Cartan solutions. R is the limit of the rings and Γ the limit of the γ_i.

The picture I want to write which is consistent with this and true, is, you have nice polynomial algebras. You have a seqeunce of maps, $k[t_1, \ldots, t_n]$ modulo k-tuple products. There are obstructions to extending the construction of Γ, but that can be rolled into d_k and the limit is $k[[t_1, \ldots, t_n]], D$. The dual of the picture is an L_∞ algebra and a limit seqence of finite dimensional coalgebras.

If I have any functor, I can represent it by starting with the tangent space potentially with differential, and use small extensions to extend it like a power series ring. This is the dual picture of an L_∞ algebra. The reason, Gabriel, you asked, if you take something other than an L_∞ algebra. This is a differential on some free object. If you have a functor of algebras of a type over O. That should be equivalent to deformations on $cobar(O)$.

Next time, I had a special request to go over the exponential. It can be talked about, a Lie algebra has an associated Lie group, and that acts on the Lie algebra. Maybe I can deal with that next time.