Deformation Theory and Operads

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Let R be a ring. Then a map $\sigma : R \to R$ is a derivation if $\sigma(rs) = \sigma(r)s + r\sigma(s)$. This can be extended to a map $\sigma : R \to M$ where M is a bimodule $\sigma(rs) = \sigma(r)s + r\sigma(s)$ for r and s in R. If R is a k-algebra, then in either case one can ask that σ be k-linear.

Here's an important example. Let p be a point in a manifold. Then you have $C_p(\mathbb{R})$ is germs of smooth $\mathbb{R}$-valued functions at the point p. This is an $\mathbb{R}$-algebra.

The second statement is that $\mathbb{R}$ is a C_p module. So for α in $\mathbb{R}$ and f in C_p, I can take $f \bullet \alpha = \alpha f(p)$. In this context then you can define the set of $\mathbb{R}$-linear derivations from $C_p \to \mathbb{R}$. These derivations are maps $X_p : C_p \to R$ satisfying $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$ and $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$. This is for f and g functions and α and β in $\mathbb{R}$. This set, according to a standard argument, is an n-dimensional vector.

Theorem 1 *$Der_{\mathbb{R}}(C_p, \mathbb{R}) \cong T_p\mathbb{R}^n$, or whatever manifold it is.*

I'm doing this because there was one point which I corrected last time about whether this was C_p to C_p or C_p to $\mathbb{R}$.

Now suppose R is a $\mathbf{k}$-algebra and R is a local algebra. Then k is an R-module, via $r \bullet \alpha = \bar{r}\alpha$. Now I can look at k-linear derivations $Der_{\mathbf{k}}(R, \mathbf{k})$ which consists of $\varphi : R \to \mathbf{k}$ which satisfy linearity and the property $\varphi(rs) = \varphi(r)\bar{s} + \bar{r}\varphi(s)$. Then the proposition is that

Proposition 1 *The $\mathbf{k}$-linear derviations of R into the ground field is naturally bijective with $hom_k(R, k[\epsilon])$ into the ring of dual numbers.*

The proof is completely straightforward, I did it last time. Given f such a linear map, $r \mapsto \bar{r} + \sigma(r)\epsilon$. Then $f(rs) = \overline{rs} + \sigma(rs)\epsilon$ whereas

$$f(r)f(s) = (\bar{r} + \sigma(r)\epsilon)(\bar{s} + \sigma(s)\epsilon) = \bar{r}\bar{s} + (\sigma(r)\bar{s} + \bar{r}\sigma s))\epsilon$$

That's just an observation. I want to remark that this $hom(\mathbb{R}, k[e]$ is a vector space. The conclusion that I want to make here is that if F is a functor of Artin rings that is representable then the tangent space to the functor, which by definition is $T_F := F(k[\epsilon])$ is a vector space.

Okay. All right, now let me say something about fiber products. Let me call them pullbacks. By the way, there's no class Wednesday. There are talks Wednesday, starting at 2PM.

So pullbacks. Given morphisms $f : A \to C$ and $g : B \to C$, we get a diagram

$$
\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}
$$

The pullback is defined to be an object $A \times_C B$ along with maps $\pi_1 : A \times_C B \to A$, $A \times_C B \to B$ so that it fits into the diagram

$$
\begin{tikzcd}
& A \times_C B \arrow[dl] \arrow[dr] & \\
A \arrow[dr] & & B \arrow[dl] \\
& C &
\end{tikzcd}
$$

The pullback should be unique among such. In the category of sets, pullbacks exist and $A \times_C B = \{(a, b) \in A \times B : f(a) = g(b)\}$. In $\mathcal{A}$ pullbacks exist while in $\hat{A}$ pullbacks may not exist.

Exercise 1 *1. Check that pullbacks exist in $\mathcal{A}$*

2. Consider this diagram:

$$
\begin{tikzcd}
k[[x, y]] \arrow[dr] & & k \arrow[dl] \\
& k[[x]] &
\end{tikzcd}
$$

Prove that the pullback of the diagram in $\hat{A}$ does not exist. (WRONG)

Given a diagram in $\mathcal{A}$ we can apply a functor $F : \mathcal{A} \to Set$ to get a diagram in Set.

$$
\begin{tikzcd}
& F(A \times_C B) \arrow[d, "\alpha"] \arrow[ddl, "F(\pi_1)"'] \arrow[ddr, "F(\pi_2)"] & \\
& F(A) \times_{F(C)} F(B) \arrow[dl] \arrow[dr] & \\
F(A) \arrow[dr, "F(f)"'] & & F(B) \arrow[dl, "F(g)"] \\
& F(C) &
\end{tikzcd}
$$

Let me do an example. Consider $F = hom(R, \bullet) : \mathcal{A} \to Set$ Let $\alpha : hom(R, A \times_C B) \to hom(R, A) \times_{hom(R,C)} hom(R, B)$.

So an element of $hom(R, A \times_C B)$ consists of maps $\varphi : R \to A$ and $\psi : R \to B$ which are compatible with, well, $f\varphi = g\psi$. This is exactly what an element of $hom(R, A) \times_{hom(R,C)} hom(R, B)$ is, it's a pair φ, ψ so that when I take their maps down to $hom(R, C)$ via post-composition with f and g, I get the same thing.

There's no reason to expect α to be a bijection in general but whon the functor is representable, you get a bijection.

Suppose $F : \mathcal{A} \to Set$. Consider the following conditions:

H0 $F(\mathbf{k})$ is a point

H2' α is a bijection for the particular diagram $A = \mathbf{k}[\epsilon] = B$, $C = \mathbf{k}$

Theorem 2 *If F satisfies these two conditions then T_F is a vector space.*

The proof is that you send $\alpha + \beta\epsilon, \alpha + \beta'\epsilon$ to $\alpha + (\beta + \beta')\epsilon$ and $\gamma(\alpha + \beta\epsilon)) \mapsto \alpha + \gamma\beta\epsilon$

You have to check that these are ring homomorphisms. [Calculation]

Now I want to apply F. This gives me a map $F(+) : F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon]) \to F(\mathbf{k}[\epsilon]) = T_F$ and $F(\gamma) : T_F \to T_F$. Because I have a functor, I have a map α from $F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon]) \to F(\mathbf{k}[\epsilon] \times_{F(\mathbf{k})} F(\mathbf{k}[\epsilon]))$. The $H0$ tells me that the right hand side is pulled back over a point, and $H2'$ that α is a bijection, so $+ : T_F \times T_F \to T_F$ is defined by $F(+) \circ \alpha^{-1}$.

Now what I'd like to do is replace $H2'$ with $H2$. Well, I need to do other things.

Definition 1 *A surjection $\sigma : A' \to A$ in $\mathcal{A}$ is a small extension if and only if the kernel of σ is a principal ideal satisfying $\ker(\sigma)m_{A'} = 0$.*

As an example, $\mathbf{k}[t]/t^3 \to \mathbf{k}[t]/t^2$ is a small extension. The kernel is generated by t^2 which is annihilated by t

Take the diagram $\alpha : F(A \times_C B) \to F(A) \times_{F(C)} F(B)$

Theorem 3 *Suppose $F : \mathcal{A} \to Set$ satisfies*

H0 $H0 : F(\mathbf{k})$ is a point

H2 α is a bijection for $A = \mathbf{k}[\epsilon], C = \mathbf{k}$

H3 T_F is finite dimensional

H4 *α is a bijection for $B = \mathbf{k}$ and $A \to C$ is a small extension.*

F is representable if and only if it satisfies $H0$, $H2$, $H3$, and $H4$.

I think I will prove this next time. The hard part is going from the axioms to representability. This was proved by Schlessinger around '63. We'll meet next Monday. I'll try to describe these ideas in a nice example of a functor.