Deformation Theory and Operads

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We meet this Wednesday, today. Monday there's no class, and then Wednesday the 10th will be cancelled. It's also possible that there will be some cancellations when the Einstein seminar is running. I don't know whether we will make things longer on some Mondays or not. So basically on these two days, I want to talk about deformations of associative algebras. This will lead into the Hochschild complex and A_∞ algebras. This course is meant to be introductory. I see some experts here. You will have to sit down and have the pleasure of listening to something that you already understand. We should be assuming no more than undergraduate algebra and undergraduate topology. Someone asked if we're going to do the signs, and I think we'll do the signs.

As an introduction, I'd like to give you a simple allegory, which I think will illustrate the goal and the techniques of deformation theory in an oversimplified setting. The goal of deformation theory is to organize mathematical objects of similar types into natural families and study variations within the families. This idea goes back, taking things, putting them in a family, and varying them, seeing how things change in the family, goes back to Poincaré at least. Here's a guiding example. Let me call this

1. Act I. A difficult problem is encountered Find the sum $S = \frac{1}{2} - \frac{1}{2(2^2)} + \frac{1}{3(2^3)} + \cdots$
2. Act II. Put the problem in a family Here I will replace the original problem S with S_t, find the sum
$$t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$
The original family was $S(\frac{1}{2})$ Though I have made this harder in some sense, now I can
3. Act III. Study variations of the problems in the family The family has some structure. If I think of t as a parameter on the real line, I can study this using the structure of where the family lives. In this case it is over the real line. In this case, I can take the derivative with respect to t, and get the expression $1-t+t^2-t^3+\cdots$. You can multiply by $(1+t)$ and get 1, so then $S'(t) = \frac{1}{1+t}$. The original problem had one trivial version at $t = 0, S(0) = 0$. So this fact, together with $S'(t)$ implies that $S(t) = \log(1+t)$. Now you've solved every problem, and can specialize your original problem by setting $t = \frac{1}{2}$.

This is a good analogy. Take a problem, make it harder by putting it in a family, but then you can use structure, and study how things vary, and then you can solve unsolved problems and see new structures.

That's just the introduction.

The first example. Deformations of associative algebras. This isn't the first example historically speaking, which was deformations of complex manifolds, which goes back to Riemann. This is the simplest example. Because this is a beginning course, I'll define an associative algebra. The references are Gerstenhaber, the cohomology structure of an associative ring, volume 78 of the annals in 1963, and on the deformation of rings and algebras. Everyone is allowed and encouraged to make remarks, they don't have to be sarcastic. This one is in volume 79, in 1964.

Fix a ring S. Commutative and associative, and unital.

Definition 1 *An associative algebra over S is a pair (A, μ) where A is an S-module and $\mu : A \times A \to A$ is S-bilinear map satisfying $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$.*

I can also say, you can say it's a ring and there's a map from S into A. You can say $\mu : A \otimes A \to A$ is a map of S-modules. I assume that A is unital; that is, there exists a $1 \in A$ so that $\mu(x, 1) = \mu(1, x) = x$ for all $x \in A$. The ring S, for the most part, will be a field of characteristic zero, and then this will be a vector space.

We say (A, μ) is commutative if $\mu(x, y) = \mu(y, x)$. The ground ring is always commutative so that I get a good tensor product, but A will not necessarily be commutative.

An algebra morphism $\varphi : A \to A'$ is an S-module map satisfying $\varphi(1_A) = 1_{A'}$ and $\varphi(\mu_A(x, y)) = \mu_{A'}(\varphi(x), \varphi(y))$. When convenient, and not confusing, I will denote $\mu(x, y)$ by $x \cdot y$ or xy. There may be various algebra structures around and I may have to give them different names.

One last thing. An augmentation of an associative algebra (A, μ) over S is a map $\epsilon A \to S$ that is a map of algebras. S is always an algebra over itself. It's just a way of mapping down to the ground ring. An example, $A = \mathbb{R}[x]/x^3$ is a three dimensional algebra over $\mathbb{R}$. The map ϵ defined by evaluation at zero is an augmentation. If you multiply two polynomials and take the constant term, it's the product of their constant terms.

Without apology, I think I'll do five more minutes of additional basic algebra. I need to talk about the graded version and complexes.

A $\mathbb{Z}$-graded S-module V is an S-module that is a direct sum

$$V = \bigoplus_{k \in \mathbb{Z}} V^k$$

of submodules.

An element $x \in V^k$ is called homogeneous of degree k. I might write $k = |x|$, which assumes x is homogeneous. A linear map $\varphi : V \to V'$ is graded of degree ℓ if $\varphi(V^k) \subset V'^{k+\ell}$ for all k.

So when you have graded modules, you have these two very important constructions that you want to look at, $Hom(V, V')$ and $V \otimes V'$, and you want these to be graded as well. So $Hom(V, V')$ admits a decomposition in terms of graded maps. $V \otimes V'$ is an S-module already. So $V \otimes V'$ of degree k is defined by $V^i \otimes V'^{k-i}$, and you sum over all is. It's like chain complexes in topology.

A complex (V, d) is a pair where V is a graded vector space and d is a map from V to V of degree 1 or -1. I'm not going to be prejudiced, satisfying the additional condition that d^2 (under composition) is zero. In this setting d is called a differential.

Given a complex there is an important construction, namely the homology. The homology of a complex (V, d) is defined to be $H(V, d) := \ker\ d/im\ d$. Because $d^2 = 0$, well, the kernel is a submodule, and the image is a submodule of the kernel, you can take the quotient, and this is also naturally graded. The kth part of the homology is the kernel of $V^k \to V^{k+1}$ modulo $Im\ V^k \mp 1 \to V^k$.

The kernel, elements in the kernel are called k-cycles, and in the image they are called k-boundaries.

Okay, the last thing that I should say as far as introductory algebra, is what a graded algebra is. So

Definition 2 *A graded algebra is a pair (A, μ) where A is a graded module and and μ is a degree zero associative algebra.*

Our first sign of the course is that a graded algebra (there are many kinds of algebras. If I don't say, I should always say associative. Out of laziness I say algebra.

A graded algebra is commutative provided $\mu(x, y) = (-1)^{|x||y|}\mu(y, x)$.

So you have to be careful whether you mean commutative in the graded or ungraded world. You have to break two arbitrary things up into homogeneous pieces.

Already I have poor pedagogy. I've already said that a graded algebra is an associative algebra and said what it means for an algebra to be commutative. So I should call it graded commutative.

[In what sense are you using the absolute value sign?]

A differential graded algebra (associative algebra) is a triple (V, d, μ) where (V, d) is a complex, (V, μ) is a graded algebra, and there's a compatibility between the differential d and the product μ:

$$d\mu(x, y) = \mu(dx, y) + (-1)^{|x|}\mu(x, dy)$$

Now let's talk about the deformations of an associative algebra. Let me talk about the intuitive idea of deforming an algebra $(A, \cdot)$ is to replace $x \cdot y$ by something like $\mu_t(x, y)$ where $\mu_t(x, y) = x \cdot y + t\mu_1(x, y) + t^2\mu_2(x, y) + \cdots$, where μ_t is associative. This is not precise because there's a change of ground ring. As Gabriel mentioned, what makes it not precise is that μ_t is not a product on A, it's got this parameter t, so let me make the definition.

Definition 3 *Let $(A,\cdot)$ be an associative algebra over a field $\mathbf{k}$ of characteristic zero. Then, a deformation of $(A,\cdot)$ over a base (B,ϵ) (where (B,β,ϵ) is a commutative associative algebra over $\mathbf{k}$ with an augmentation $\epsilon : B \to \mathbf{k}$) is an associative algebra structure μ on $A \otimes_{\mathbf{k}} B$ over B such that $A \otimes B \overset{canonoid\otimes\epsilon}{\rightarrow} A$ is a map of algebras.*

In Gerstenhaber's paper, he does one example, where the ring is $k[[t]]$, which is the same in the commutative or noncommutative sense. No one asked what to do with a graded base space. Many tools require the use of commutative base spaces. In terms of geometry, in a geometric picture, where I have spaces and points in the space, I have the vector space, A_t fibered, then I want B to be the space of functions, which are commutative. There's a whole noncommutative geometry so you probably want to let these be noncommutative.

[One might want to apply algebraic geometric methods to deformation theory?]

Much of this has been done by Grothiendieck, others

[Where are the dual numbers?]

That's a particular kind of deformation, it will be called infinitessimal.

Let me do an example. If $B = k[t]$ (this is a nice base ring, the ring of functions on the affine line), then you have the nice augmentation by evaluation at zero. I'm taking the maximal ideal corresponding to zero and modding out, which is the augmentation ideal for this map. I get the ground field, I almost want to require x to be algebraically closed so that you don't get an extension. So then what does it mean? Then $A \otimes_k B$ is isomorphic to (as a vector space) $A \oplus tA \oplus t^2A \oplus \cdots$, so this will be polynomials in t with coefficients in A. Then μ will be a map from $A \otimes B \otimes_B A \otimes B \to A \otimes B$ which is linear over the polynomials. I only need to define this on a B-basis of this space, which is A. So I can just take $\mu(x,y)$, where x comes from A. The result will be some polynomial in t, so it will look like $\mu_0(x,y) + t\mu_1(x,y) + t^2\mu_2(x,y) + \cdots$ For a particular x and y it will terminate, just be a polynomial.

Okay, now this is the basic setup and now I want to assume that the map from $A \otimes B \to A$ is the projection that kills t, sets $t = 0$. So then if the map $A \otimes_k B \to A$ is a map of algebras (over $\mathbf{k}$), that is equivalent to saying, if I multiply and then set $t = 0$ I get the constant term of μ; in the other order I get $x \cdot y$. So $\mu_0(x,y) = x \cdot y$. This captures the full range of Gerstenhaber's original idea. This corresponds to a sequence of bilinear maps so that the new thing is associative.

The reason I'm using polynomials is because you have evaluation maps. So $B = k[t]$ has evaluation maps ϵ_α for each $\alpha \in k$, where $\epsilon_\alpha(p(t)) = p(\alpha)$. For power series you wouldn't have evaluation maps. But just to keep things simple, you have these evaluation maps. You can think of these as being different augmentations. For each α you have the maximal ideal $t - \alpha$ modding out by which gives you a field isomorphic to the ground field.

You can define, given a deformation μ of $(A, \cdot)$ over $B = \mathbf{k}[t]$, you can compose with evaluation maps and define new associative products on A; call them $\cdot_\alpha$ for each $\alpha \in \mathbf{k}$ using the evaluations.

So this is $\epsilon_\alpha \mu(x, y) =: x \cdot_\alpha y$ for $x, y \in A$.

So you get the picture, above $\mathbf{k}$ the affine line, of a family of algebras fibered over this line. Over 0 you have the original product $\cdot$. That's guaranteed by saying that the original augmentation is a map of algebras. So you get a family of algebras over the real line.

I could have started by defining deformations of algebras over spaces. I could use other base rings, like $k[t_1, \ldots, t_n]$; if I used this, for every point in $\mathbb{R}^n$ I would have an algebra. I could take, if I can give up my attachment to geometry, is choose power series rings in one or many variables. Then deformations over $\mathbf{k}[[t]]$ are called formal one parameter deformations, and $k[[t_1, \ldots, t_n]]$ are formal n-parameter deformations. Here I would allow infinite power series in t, I keep the same picture, and imagine I have evaluation maps, although I don't have them. In algebraic geometry, as a scheme, you have one point over zero, where you have an evaluation map. You keep the picture in mind even if the polynomials are infinitely long.

The word formal refers to formal power series. If $\mathbf{k}$ is a topological ring, you could do something like $\mathbb{C}\{\{t_1, \ldots, t_n\}\}$. I would mean, for the next ten seconds, has a positive radius of convergence. But I'm not using these rings. Here's a good ring: $\mathbf{k}[t]/t^2$. So deformations over this ring are called infinitessimal. If you've seen these before, they might just call this $k[\epsilon]$ and everyone knows that $\epsilon^2 = 0$ because it's so small.

The use of base rings is what unites the subject. If you have deformations of complex manifolds, vector bundles with connections, associative algebras, group representations, you organize it as families over a space, which gives you the idea of using base rings. It's functorial so looking at functors of base rings gives you things all over the board.

I just have three minutes left, so let me do an example. Everyone in here will do this exercise. If you take a formal one parameter family of deformations, you write down what it means to be associative, and you get a list of conditions that these things need to satisfy, that's a good place to open up the next lecture.

So let's look at $\mathbf{k}[x]/x^3$. Then we can make a table

	1	x	x^2
1	1	x	x^2
x	x	x^2	0
x^2	x^2	0	0

of multiplication. Consider $B = \mathbf{k}[t]$ and define μ by, well, I want the answer to be $\mathbf{k}[x, t]/x^3 - tx$. This is isomorphic to $\mathbf{k}[x]/x^3 \otimes_{\mathbf{k}} \mathbf{k}[t]$ As a $\mathbf{k}[t]$-module. The former has a nice associative product. You can see in terms of the table:

	1	x	x^2
1	1	x	x^2
x	x	x^2	tx
x^2	x^2	tx	tx^2

So I'm taking a product that was defined and deforming it in the t direction. That's an example to keep in mind. It's sort of a free algebra on one relation, so that instead of $x^3 = 0$ we have $x^3 = tx$.

For your first homework exercise, are A_1 and A_2 isomorphic. Is A_t isomorphic to A_0? If they're all isomorphic it's not very good, you've arranged your algebras as a fibration over the real line. So it's not very interesting if the answers are yes.

For homework two, determine the condition on μ_1 implied by $\mu_t(x, y) = xy + t\mu_1(x, y) + \cdots$ being associative.