# Focused Research Group Workshop 

Gabriel C. Drummond-Cole

November 15, 2008

## 1 Jacob Lurie IV

There is a construction which starts with a category $C$ and produces a simplicial set called the nerve of $C$. Take $N(C)_{n}$ to be the sequences of $n$-composables. This is a simplicial set encoding the structure of the category. You can sort of think of a category as a simplicial set satisfying conditions. The world of categories sits inside the world of simplicial sets (classes). So the theory of complete Segal spaces is an attempt to replace categories by $\infty-1$ categories and simplicial sets by simplicial spaces. So the relationship is similar to that idea.
[I wanted to know how that leads to a definition or an example of $\infty-n$ category.]
You might ask, which simplicial sets arise as the nerve of a category. A simplicial set $X$ is equivalent (isomorphic) to the nerve of a category if and only if for all $n \geq 0$, the canonical map you can write down $X_{n} \rightarrow X_{1} \times_{X_{0}} X_{1} \times_{X_{0}} \cdots X_{1}$. Let me draw the picture Dennis drew. An element of $X_{3}$ is encoded by a 3 -simplex, which says the 3 -simplex is determined by the three edges that span all the vertices of that simplex.

Now make this a definition: A simplicial space $X$ is a Segal space if for all $n \geq 0$ the map $X_{n}$ to the $n$-fold fiber product of $X_{1}$ over $X_{0}$ is a weak homotopy equivalence.

You also want to require that the invertible morphisms come from $X_{0}$. So an $\infty-1$ category is a Segal space plus an extra condition. This is either a definition or you could give another definition and this would be a theorem. This extra condition is called completeness and was introduced by Rezk.
[In technical terms, [unintelligible], whenever you have a [unintelligible]that is [unintelligible]to complete Segal spaces, [unintelligible]complete Segal spaces?]

I don't think it says that. Suppose you have [unintelligible]and then an interval, a cosimplicial object of $A$, then $A$ is equivalent to complete Segal spaces if and only if, and there is a list of conditions. The most important encodes the weak homotopy equivalence statement.

Let me state this differently. Let me work in the language of $\infty-1$ categories. An example of an $\infty-1$ category is the collection of $\infty-1$ categories. That's an $\infty-2$ category but let's
not go that far. In the world of $\infty-1$ categories, what can you say about this. You have the ordinary category of simplices $\Delta$ and you can map this to $C a t_{\infty, 1}$. This takes $n$ to the category $0 \leq \cdots \leq n$. This is universal with respect to the properties: it admits small (homotopy) colimits, $[n]=[1] \sqcup_{[0]}[1] \cdots[1]$ and something like the "two out of six axiom," which is a translation of the completeness axiom, which I haven't explained precisely. The picture is if you take this [picture] and collapse these two edges to a point, taking $* \sqcup_{\{1,3\}}[3] \sqcup_{\{0,2\}} * \cong[0]$.
[In the $\infty-0$ world gives you infinity spaces with symmetric monoidal. Do you have an analogue in $\infty-1$ ?]

It's an $E_{\infty}$ object in the same sense. You can say what a symmetric monoidal $\infty-1$ category is, and then there's an obvious notion of the product of $\infty-1$ categories. In terms of the Segal spaces this can be realized just as $\infty-1$.

If I wanted $\infty-n$ I would say that it still admits small homotopy colimits, I would extend the $* \sqcup[n] \sqcup *$ in all directions, and the axiom three would be a little more complicated.

Complete Segal spaces is rigged to have this universal property. It's a bisimplicial space that force these properties. If you have a cosimplicial object in another category, and you can make a left Quillen functor. You'll be able to factor this through complete Segal if it satisfies axioms and if it satisfies further axioms it's an equivalence. That was yoga for showing that different models give equivalent theories.
[How new is this? In some sense it seems quite new because you're extending homotopy theory to a wider set of objects, but in the other sense it's homotopy theory, like turtles, all the way down, is there a way to say that this is a homotopy theory of homotopy theories?]

Well, I don't,- [As an old-fashioned country homotopy theorist, this brings some of the terms of category theory, so instead of looking at paths you look at something with an adjoint, and a dual. Now you want to enlarge that, you have an adjoint which is a new notion. It's the impact of those kind of concepts.]
[Which things we know from homotopy theory generalize to this? Postnikov towers?]
Yes, well, let $C$ be an $\infty-n$ category with $m \geq n$. I'll define $\tau_{\leq m} C$, which is an $(m, n)$ category: above $n$ morphisms are invertible and above $m$ the identity. The objects are objects of $C$. The 1-morphisms in $\tau C$ are 1-morphisms truncated, $\tau_{\leq m-1} 1 H_{o m}(X, Y)$. Provided you're keeping the $\pi_{0}$, the connected components of a space and its truncation are the same. You have to understand the loop spaces of the truncations, which are (shifted by one) truncations of the loop spaces. This is a definition of what it means to truncate something.

In $\infty-0$ categories, how do you characterize $X \rightarrow \tau_{\leq n} X$ ? The homotopy groups $\pi_{i}\left(\tau_{\leq n} X\right) \cong$ $\pi_{i}(X)$ for $i \leq n$ and is otherwise trivial. If I look at $\Omega\left(\tau_{\leq n} X\right) \cong \tau_{\leq n-1}(\Omega X)$. If I wanted to be more precise, I would view this not as a construction but a condition that I can put on a map, a map exhibits $Y$ as a Postnikov system for $X$ if these conditions are satisfied.
[Can you describe the fibration of $n+1$ to $n$ in an elemental way?]

Say that $C$ is an $\infty-n$ category. You can truncate it all the way to get the homotopy $n$-category of $C$, ignoring all the larger morphisms. This is just $\pi_{0}$ of my space. In classical, you have $\pi_{0}$ which is a set, then $\pi_{1}$ is possibly non-Abelian so complicated, and then above you have principal fibrations. This is non-Abelian cohomology, you have to do things by hand. You have to deal with it, you get one argument that takes care of the higher parts and then a much harder version for $\pi_{1}$.
[missing]
If $n=1$, a "local system" is the same thing as a functor from the arrow category of $C$ to Abelian groups. A morphism $f \rightarrow f^{\prime}$ is a factorization of $f^{\prime}$ into pieces one of which is $f$. Given such a thing you can make an Eilenberg-MacLane space which is fibered over $C$.

Maybe I can say a little bit about what goes into the proof. Let me remind you what the theorem says:

$$
\operatorname{Fun}^{\otimes}\left(n \operatorname{Bord}^{G}, C\right) \cong\left(C^{f d}\right)^{G}
$$

You might as well assume that $C$ is already fully dualizable. Next you can reduce to the case where $G=O(n)$. Remember that's the case of unoriented manifolds. It probably seemed before that the basic case was that $G$ was a point. To give a proof, the universal case is the unoriented case. Every framed manifold is an unoriented manifold. Knowing something about the unoriented manifold would tell you something about the framed setting, but vice versa would be harder. Since I reduce to $G=O(n)$, I haven't proven the framed case yet. So let me reformulate inductively. We saw some of this. In the previous lecture I took $n=2$ and started with the $n=1$ case and then studied what it took to promote this. You reformulate this statement in terms of what it takes to promote, assume that you have a field theory $Z_{0}:(n-1)$ Bord $^{u n} \rightarrow C$ and you want to produce an extension to $n B o r d^{u n}$, assuming we already know what an $n-1$ field theory is.

Given $Z_{0}$, in particular, you can take $Z_{0}\left(S^{n-1}\right)$, which is an $n-1$-morphism in $C$. Just as yesterday, this has an action of anything that acts on the $n-1$-sphere, in particular an action of $O(n)$. Extending (this is a claim) $Z_{0}$ to an $n$-dimensional field theory $n B o r d \rightarrow C$ is equivalent to giving $\eta$ (secretly $Z$ on the $n$-disk) which is in $\operatorname{Hom}\left(Z_{0}\left(S^{n-1}\right), Z_{0}(\emptyset)\right)$. You'd expect this to be an $O(n)$ fixed point, so it is an $O(n)$-fixed such $\eta$ such that some finiteness condition is satisfied. The finiteness condition is something like being the counit of an adjunction. The next step is to do what I just erased, examine the Postnikov tower of $C$. We have a map $(n-1)$ Bord to the inverse limit of the Postnikov tower, and we want to extend it to nBord given $\eta$. Suppose we could do this at level $n$. Now we can try to see what it would take to lift this. Going to $\tau_{\leq n} C$ is not good enough, you want to start one level up because of non-Abelian $\pi_{1}$. From then on you can describe things cohomologically. Lifting a map along such a fibration is described by cohomology of the source with coefficients in the Abelian group objects defining the Eilenberg MacLane spaces. So these liftings are controlled by cohomology. You have to do some sort of, well that's a linear problem, it's a calculation you have to do in stable homotopy theory. You do the computations and show that the obstructions to lifting are precisely those to generating $\eta$. This is analagous to Galatius-Madsen-Tillman-Weiss.

Step six is to do the hard work, now the hard work, which is the case where you can replace $C$ by the truncation $n+1, n$ category. Now this is tricky because we're dealing with $n$ complicated, which are tricky. It's less complicated when most morphisms are invertible. Reduce this to a problem about $\infty-1$ categories. I'm being vague, I know. This is the main place where you use the existence, the fact that $C$ has duals. Once you've reformulated this, since you've reduced to the case without many higher morphisms, you can understand this as a problem of describing a $(2,1)$ category of bordisms between $n-1$-manifolds with boundary. The kind of thing you reduce to has 2 -morphisms which are diffeomorphisms up to isotopy between bordisms (along with bordisms of the boundary). Next you study these using Cerf theory. To understand such a category, you want to understand the mapping spaces, so you're looking at essentially $(1,0)$-categories, trying to understand some groupoids. What do they look like? If you fix the boundary of an $n$-manifold, you think of the groupoid of manifolds bounding a given manifold and diffeomorphisms between them up to isotopy. One way to attack this is by choosing a Morse function on this manifold. Doing this, if the manifold gives you a bordism, this gives a composition of bordisms of a very simple type. That, classical Morse theory gives you something that surjects onto $\pi_{0}$. Anything with a given boundary can be built by handle attachments. You need to understand $\pi_{0}$ exactly and then $\pi_{1}$ exactly. With Morse theory, you need generic one and two parameter families of functions, which are not Morse sometimes, and you need to study families of such functions. This is well-understood when the dimensions of the families are low. All the singularities that arise have natural categorical interpretations. In particular, you can understand why you can see all of those singularities in terms of the assumption that $C$ has duals. These appear because certain morphisms have adjoints, the unit and counit are compatible in a certain sense, and that is a very vague outline.
[Can you give more detail about how the duals come in in step six?]
Yeah. Let me stick to two dimensions. We have the goal to understand an ( $\infty-2$ )-category $C$ assuming $C$ is symmetric monoidal and every object has a dual in the naive sense. These are complicated, so let's try to understand in terms of simpler objects. So start by obtaining an $\infty-1$ category $C_{0}$ by discarding noninvertible 2-morphisms. So $C_{0}$ is a lot like $C$. Then remember those, but in a clever way. Note that to understand $\operatorname{Hom}_{C}(X, Y)$, because $X$ is dualizable, this is $\operatorname{Hom}_{C}\left(1, X^{\vee} \otimes Y\right)$, so what you need to understand is $Z \mapsto \operatorname{Hom}_{C}(1, Z)$, this functor. What can you say about that functor $(F)$ ? It's a functor defined on the $\infty-1$ category $C_{0}$, and takes it to the $\infty-1$ category $C a t_{\infty}$. This is lax symmetric monoidal, so there is a map $F(X) \times F(Y) \rightarrow F(X \otimes Y)$ but not an equivalence. Now we can reconstruct $C$, The objects we know, and now we can reconstruct $\operatorname{Hom}_{C}(X, Y)$ as $F\left(X^{\vee}, Y\right)$, and now we can understand $F$ as a fibration $p$ of $\tilde{C}_{0}$ over $C_{0}$ so that $F(Z)=p^{-1}\{Z\}$. Now the condition that $F$ is lax symmetric monoidal gives a symmetric monoidal structure on $\tilde{C}_{0}$ and we have converted $C$ into a diagram. Any questions about $C$ can be rephrased to be a question about this fibration.
[Where is composition?] Suppose I want $\operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \rightarrow \operatorname{Hom}(X, Z)$. The domain here is $F\left(X^{\vee} \otimes Y\right) \times F\left(Y^{\vee} \otimes Z\right)$ which maps by the lax monoidal property to $F\left(X^{\vee} \otimes Y \otimes Y^{\vee} \otimes Z\right)$ which then maps by evaluation to $F\left(X^{\vee} \otimes Z\right)$. So it comes from the lax monoidal structure.
[What are the extra properties?]
Well, $C_{0}$ must have duals, in the naive sense. $\tilde{C}_{0} \rightarrow C$ is a cocartesian fibration. Then the collection of coCartesian morphisms in $\tilde{C}_{0}$ are stable under duals. The underlying left fibration of $p$ is equivalent to $\left(C_{0}\right)_{1 /} \rightarrow C_{0}$. The $1 /$ means objects which receive a map from 1.
[Can you choose a category $C$ for $n=2$ like Frobenius algebras or something? Let me say it, well, you have to do the cocompact version and pretend, what's the discussion of the thing for string topology if this were the relevant version.]

Yesterday we had the $\infty-2$ category with objects differential graded algebras, bimodules, and maps of bimodules. (You need to twist to do nonzero dimension)
[What are the fully dualizable objects, twisted?]
These are "smooth and proper dgas" which means $A$ such that $A$ is dualizable over $k$ and over $A \otimes A^{o p}$. For string topology, you want either cochains on a manifold or chains on the based loop space. The chains of the based loop space is dualizable over itself but not over $k$; the cochains on a manifold are precisely the reverse.

Let me give an example in the holomorphic setting. Let $X$ be a smooth projective variety over $k$, let $\mathscr{E}$ be a "generator" of the derived category $D_{c o}^{b}(X)$ of coherent sheaves in $X$. Now you can look at the endomorphisms of $\mathscr{E}$, resolve it, and organize this into a differential graded algebra over $k$. This is an example of a smooth and proper dga's. The fact that it is smooth and proper comes from $X$ being smooth and proper (from projective). Proper gives you over $k$; it means compact. Smooth gives you $A \otimes A^{o p}$ To do something analagous in higher dimensions, there would be [unintelligible]conditions here. There would be three conditions in three dimensions which are very hard to satisfy simultaneously.
[Is there an example for the $A$-model?] Put in Fukaya categories, but I don't know what that is.
[Physicists talk about string theories, they want a differential form on the moduli space of Riemann surfaces. This is an algebraic topological gadget, it has values in some Hilbert space. It's a cochain, translated here, and that's exactly what it is in one of these field theories. If you rationalize the higher category things, it's maps from the moduli space of Riemann surfaces, with values in the endomorphisms of the vector space you assign to the circle.]
[The physicists would say that there are homotopies that tell you it doesn't depend on the conformal class of the metric, starting from 0 . They construct a particular cochain in some degree. They will see the real homology of moduli space. They only care about the top chain.

