Focused Research Group Workshop

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November 15, 2008

1 Jacob Lurie

So let me begin with $\infty - n$ categories. This is something which has objects X, Y, Z, morphisms $1Hom_{\mathscr{C}}(X,Y)$, two morphisms between the one-morphisms, along with associative composition for morphisms and all k-morphisms invertible for $k > n$. For example, an $\infty - 0$ category is an ∞-groupoid, which is something like a homotopy theory. You think of the objects as points, the two-morphisms as homotopies between paths, and so on.

Let me introduce now the main example about which I want to talk, which is $nBord$. The objects, well, let me introduce a convention. All manifolds are smooth, compact, and might have boundary or corners. The objects in $nBord$ will be 0-dimensional manifolds. The 1-morphisms, if I have two, I can consider bordisms between them. These are then given by bordisms, one dimensional manifolds with boundary. Similarly the two-morphisms are given by bordisms between one-morphisms, bordisms of bordisms, which are encoded by two-dimensional manifolds with corners. Now I will write .... The n-morphisms of $nBord$ will be n dimensional manifolds with corners, and the higher you get in dimension, the more complicated things get. One way to make this precise, you don't compose on the nose, but specify the relation that says that h is a composition of f and g. You care that there is a contractible choice of compositions, rather than one particular one. Then $n+1$-morphisms are diffeomorphisms, $n+2$-morphisms are isotopies of diffeomorphisms, and so on. Let's have some variants, like $nBord^{un}$ where they're unoriented, $nBord^{or}$ where they're oriented, $nBord^{Spin}$ where you use Spin manifolds, or $nBord^{fr}$, where these are framed manifolds. All of the k-manifolds will have a trivialization of their tangent bundle sum $\mathbb{R}^{n-k}$.

This is, what you might call a symmetric monoidal $\infty - n$ category. The tensor product is disjoint union of manifolds.

Now let me introduce the main definition. An extended n-dimensional topological quantum field theory with values in C is a tensor functor from $nBord$ into C. Here C is another object which has the same nature as $nBord$, a symmetric monoidal $\infty - n$ category. For example, you could take C to be the ordinary category of vector spaces. The higher morphisms aren't anything except the identities. In that case, that's an $\infty - 1$ category, so you can talk about 1 dimensional quantum field theories in that category. The classification of these is easy.

A one dimensional topological quantum field theory is specified by a vector space, it's the vector space associated to a point.

Theorem 1 *(Baez-Dolan Cobordism Hypothesis)*
Mike probably stated this for you; $Fun^{\otimes}(nBord, C)$ *(let me first state this in the framed setting) is equivalent to the subcategory* C^{fd}*, the* ∞*-groupoid of fully dualizable objects in* C*. This is an equivalence with a natural direction from the left to the right, a map which sends* Z *a TQFT to* $Z(*)$*.*

Before I say any more about this, maybe I should just say something about this. Let me say a few words about why you might believe this. Let's consider as a warmup the case $n = 1$. Let's suppose $Z : 1Bord \to C$. To understand Z you might start by applying this to objects. This is the n-framed version. That's an orientation here. So here the objects are points, some with positive and some with negative orientation.

You can evaluate on the empty set $Z(\emptyset)$ will be the unit object $1 \in C$. You could pick a point with the positive orientation, and so $Z(\overset{+}{*})$ gives $X \in C$. You can also act on a negatively oriented point, but this gives you nothing new. You can evaluate your field theory Z on a 1-dimensional bordism from two points with opposite orientations to the empty set, the unit. Similarly, I can evaluate Z on the bordism from nothing to two points.

These maps are compatible in some sense and they exhibit X and Y as "duals" in the category C. Giving a map $X \otimes Y \to 1$ is like a bilinear map $X \times Y$ to the unit, which is like a map $X \to Y^*$, and you learn that this is an isomorphism and both of these are finite dimensional (in the vector space category, at least). So X and Y are duals which satisfy some finiteness condition. You can still formulate the finiteness condition. This is formulated in terms of the existence of a dual satisfying a finiteness condition.

[Uniqueness?] So it only makes any sense here to consider uniqueness up to a contractible set. To summarize, specifying Z on a negatively oriented point is just a finiteness condition. It says that there is a dual, and Z on the negative point gives the dual. Now you know what Z does on a zero manifold. You see that 0-manifolds are made up of the disjoint union of positive and negative points.

So now for morphisms, you only need to work with those morphisms which are connected. There are only five, the identity $X \to X$ and $X^{\checkmark} \leftarrow X^{\checkmark}$, the duality maps $X \otimes X^{\checkmark} \overset{ev}{\to} 1$ and $1 \overset{coev}{\to} X \otimes X^{\checkmark}$. I can break up the circle, and see that it is the composition of the evaluation with the coevaluation, which is the dimension of X. So dim $X \in 1Hom_C(1, 1)$. This data tell you that whatever this object is, once I identify with the dual, this becomes evaluation and coevaluation. Better to call it X and Y, and say that the whole package is the same as giving X.

[All of the things we know from the language of categories. Can you add some infinity?]

When $n = 1$ this is more or less obvious. You can start building a field theory if you are given a finite dimensional vector space X. Working at the level of ordinary categories, you can turn that into a proof, but if the target is a higher category, there are more things that you need to do. Let me illustrate by showing a phenomenon that you hit even in the $1Bord$ case. Now 1-manifolds don't have much of diffeomorphisms, but for example there are rotations. So $1Hom_{1Bord}(\emptyset, \emptyset)$ is the same as the classifying space for closed 1-manifolds.

Every one-manifold is a disjoint union of circles, so this breaks up into classifying spaces of n circles. Inside this I have the connected component corresponding to the case where I have one circle, so I have $BDiff(S^1)$ which is $\mathbb{CP}^\infty$. If you had n-circles, you'd have the homotopy quotient $(\mathbb{CP}^\infty)^n/\Sigma_n$. So now, anyway, what I'm trying to convince you of, you have morphisms spaces, they're not so complicated but they're not discrete.

Now suppose you have a functor $1Bord \to C$. Then you have maps $1Hom_{1Bord}(\emptyset, \emptyset)$, I can see what this does, and it will go into $1Hom_C(1, 1)$, which will be a map of $\mathbb{CP}^\infty$ into here. That's the dimension of X which is a single point. That's what this map does to a basepoint of $\mathbb{CP}^\infty$, because I broke up the circle into a composition of two things. That *hom* was supposed to have a circle action. I can't break a circle up equivariantly, so I broke this up and only saw what happened to the basepoint. When you have a dualizable object you get a field theory. You also know that the dimension of $\mathbb{C}$ has a circle action, a natural extension to all of $\mathbb{CP}^\infty$, so this will be, in the vector space case. If you worked with complexes, you'd get a complex of sheaves with locally constant cohomology. This predicts that the dimension of X should have a canonical circle action. You might ask why not take all of $\mathbb{CP}^\infty$ to this constant point. What the map is supposed to be is uniquely determined by X and is usually not a constant map if X is a higher catogory.

Let me give an example. C will be an $\infty - 1$ category whose objects are commutative rings. The 1-morphisms in C will be $1Hom_C(A, B)$ will be complexes. I should have fixed k, and had my rings be associative k-algebras, not necessarily commutative. I could write morphisms to be given by bimodules. I want them to be given by complexes of (A, B)-bimodules. Then 2-morphisms will be quasiisomorphisms between chain complexes. 3-morphisms are chain homotopies. The punchline is that everything is dualizable, and the dual of A is A^{op}. I should say that this is symmetric monoidal with tensor product over k. If I take A I should be able to write down interesting maps $k \to A \otimes A^{op}$ and $k \leftarrow A \otimes A^{op}$. So there's a natural thing to put in both of these, and that's A. These give me one-morphisms in this category, and give A^op as the dual of A. You compose by taking tensor products, but those have to be left derived. Let's compute the dimension of A. This should come from composing the coevaluation map with the evaluation map. So this should be a morphism from k to itself.

So A is a right and a left module over $A \otimes A^{op}$ So I should resolve on the left and get $A \otimes^L_{A \otimes A^{op}} A$ is a chain complex of k-vector spaces. By definition, this is a chain complex which computes the Hochschild homology of A, $CH_*(A)$. This has a circle action, which is something that you see classically if you construct this carefully enough. You also get the circle action from this general result, and in general it's nontrivial.

[What does it mean if it is?]

I don't know.

So that the circle action is nontrivial tells me that there is more work to be done to produce the map knowing just the object. If you thought about this calculation, what if I broke this into n pieces along with cyclic symmetries, you could see that you had some cyclic object whose geometric realization was giving you this.

[In the case that this is your category C, then the space of these topological field theories corresponds to the set of As and?]

It would be a classifying space of As, but the theorem would say that the circle action comes for free, that you just have to specify the A.

Let's believe the theorem in the general case, focusing on this bogus reason, you take an arbitrary manifold and decompose it into simple enough pieces; since Z is a functor its action on a complicated manifold should come from what it does on its simple enough pieces. If you know what Z is doing on the small bits from what it's doing on a point, that's enough. Why should you be using framed manifolds? Every manifold, locally, looks like its tangent space at a point. So what you have more canonically is that you can break a manifold into simple pieces the simplest of which look like the tangent space at a point.

Let me rewrite the Baez-Dolan conjecture. You should have a statement like this in the framed case, $Fun^{\otimes}(nBord^{fr}, C) \cong C^{fd}$, and this has an action of $O(n)$, the automorphism group of the trivial bundle. So you have an action $O(n)$ on the space of fully dualizable objects.

In the example where $n = 1$ $O(1)$ has two objects, there is a natural action on the space of dualizable objects, which is dualization. Say you have a symmetric monoidal $O(n)$ action. So $O(n)$ has two components. I will have a fully dualizable objects. I have a map from $O(n)$ into the fully dualizable objects. I will have the original object and its dual from the components. When $n = 2$, if X is a fully dualizable object, you get a map $O(2) \times \{X\} \to C^{fd}$, so you have a map $S^1 \to C^{fd}$. The basepoint goes to X. Extending that says you have a certain automorphism of X. I don't think that, depending on where these lectures go, I may say more about what this automorphisms turns out to be.

[Is all of this compatible under reducing n?] Absolutely.

Let me give another example in which you see this $O(n)$ action in which it reduces to something concrete that you will be familiar with. Let's take C to be an ∞-groupoid, i.e., a space X. We want symmetric monoidal structures, so a symmetric monoidal ∞-groupoid. So you have a commutative associative multiplication. So you have an E_∞ space. Now further assume that every object of C is fully dualizable. If you think about what this means in this setting, you see that π_0 is commutative and associative on the nose, but this should be a commutative group, that you have inverses. Now homotopy theory will tell you that such a space is an infinite loop space, that is, the zeroth space of a spectrum. So you can write $X = \Omega^n Y(n)$ for all n.

Now an infinite loop space is the same data as a cohomology theory. So this is the same as giving a cohomology theory. So what's the set of fully dualizable objects of C? It's the same, and so C^{fd} is just X. Now X has an $O(n)$ action and they are compatible with one another. This comes from X being an n-fold loop space, $X = map(D^n, S^n), (Y, *))$. So these reduce to the J homomorphism in algebraic topology. I don't see how to make more appear from the field theory point of view. The J homomorphism sits in the diagram:

$$O(n) \to \varinjlim O(n) = O(\infty) \to GL(S) \text{ which acts on } X$$

Let me restate the Baez Dolan conjecture a different way. The framed version of $nBord$ is the free symmetric monoidal $\infty - n$ category on one fully dualizable object.

Let me talk about this hypothesis in the non-framed version. Suppose I have some group G mapping to $O(n)$. Then I can define a G manifold as a manifold whose structure group I can reduce to G. When G is a point, $nBord^G$ is the framed version. When G is $SO(n)$ it's the oriented version of the bordism category. The statement you would make in this context is that if you look at tensor functors $Fun^{\otimes}(nBord^G, C)$, well, you can evaluate on a point, so that point is naturally a homotopy fixed point with respect to the homotopy action of G. So this goes to $(C^{fd})^G$. What does this mean in the simplest possible example, with $n = 1$ and $G = O(1)$. So I'm dealing with unoriented bordisms. The fully dualizable objects of C, I want the invariants under the involutions to the dual. So this is the space of self-dual objects. That equivalence should be equivariant with respect to this action, so it should be symmetrically self-dual.

[Is there a category where the symmetrically self-dual objects are the closed manifolds?] I don't know how to do that. Break? Questions?

2 Lurie, II

Apparently there will be two more of these. I don't really have a plan. Maybe I should take a survey of what people would like to hear about. Maybe I'll take a survey. I formulated the Baez Dolan hypothesis. Going forward, we could spend some time talking about definitions, like what is an $\infty - n$ category. Another thing that we could talk about is how to prove the assertions. Third, we could illustrate this with examples. We could talk about generalizations to manifolds with singularities. This might not look like as much fun as it is in practice. We could also talk about generalization to the tangle hypothesis, where the bordisms are embedded in a Euclidean space. This, you see things like, well, one special case is where the Euclidean space is dimension three and you're thinking about one-manifolds, and you're talking about tangles. These are some possibilities.

[We have three more talks. Let's do the first three but not in that order, filling things in as needed.]

Why don't I talk about string topology. So thwo dimensional theory in $G = SO(2)$. So $Fun^{\otimes}(2Bord^{or}, C) \cong (C^{fd})^{SO(2)}$. What does it mean for X to be fully dualizable. First it means that there exists a dual for X. It should be dualizable in the ∞_1 sense, subject to some compatibilities. You can say this without mentioning 2-morphisms, it only depends on the ordinary category you get by ignoring the higher parts of the $\infty - 1$ category from truncating the $\infty - 2$ category. Next you should demand for finiteness that the evaluation and coevaluation have left and right adjoints, as many as needed. Let me say a little bit about how all this works. Consider the coevaluation map. Imagine this has a left adjoint, some map $(X \otimes X^{\vee}) \to 1$ If I look at one-morphisms to the unit, I can move $Y^{\vee}$ on the left into Y on the right. So these are 1-morphisms $X \to X$ and I get such a map S, which is so called for Serre functor. If finiteness is satisfied you get a natural map from X to itself. The identity corresponds to the evaluation map. So S is the identity if the left adjoint to the coevaluation is the evaluation.

I should remark, let me make a claim, if X is fully dualizable then S is invertible. How do I think about this? The first thing is that given a fully dualizable object X, we get this natural one-morphism from X to itself. So inside the space of objects I have a point that corresponds to X, and I have a loop in that space, a one morphism $X \to X$. That loop is a map from a circle into the space of fully dualizable objects C^{fd}. This is $SO(2)$, and S is implementing this action. So there is a map $SO(2) \times M \to M$. What does this mean? It means that given a point $X \in M$, I have a map $SO(2) \times \{X\} \to M$. So the basepoint goes to X and I have a loop that starts and ends at X. Applying that in this context, you get an automorphism which is precisely defined in this way. S is the circle action.

How to think about S? It's what you get when you unwind the statement that X has a circle action. Second, you can look at an example, let's look at $2Bord^{fr}$. The fully dualizable object is a point. Everything has to be framed in dimension two. This comes with a framing (orientation preserving or reversing). It's ambiguous to write $X^{\vee}$, but that you have an evaluation map $X \otimes X^{\vee} \to 1$ that exhibits $X^{\vee}$ as a dual to X. Before we only had orientation in dimension one. Now we have orientation in dimension two. You might ask which one of these gives the right evaluation map, but that's wrong. There are maps $X \otimes Y \to 1$ and the question is, do these give an identification of Y with $X^{\vee}$. We should instead of calling Y the dual, keep track of the map. There are an integers worth of them.

To think about what S is, we'd have to think about which evaluation map to use. There are coevaluation maps $1 \to X \otimes Y$ and the same thing is true. These come in pairs, so they match up exactly. We have the coevaluation map. Does it have a left adjoint? It's given by an evaluation map by a shift. That twisting, well, what do those look like? The bordisms from X to X, considering framings, we see that there is an integer's worth of possibilities. Here's an example of a 1-morphism in the framed one-category. Up to a sign, they differ by 1. The right adjoint differ by 1 in the other direction.

Let me give a reformulation of the Baez Dolan result for $n = 2$ and $G = SO(2)$. What are you trying to answer? What does it take to make a tensor functor Z from $2Bord^{or} \to C$. The first thing that you need to do is give a tensor functor from $1Bord^{or} \to C$. Now by the previous result, this is the same data as a dualizable object $X \in C$.

So there's a dual object, a coevaluation map, and an evaluation map, and you can compose them and get the dimension of X. What else do we expect? This map S should be trivial. What's that saying? The coevaluation, the evaluation should be left adjoint to the coevaluation.

We expect that the left adjoint of the coevaluation be the evaluation. Then $coev^L \circ coev$, by the definition of an adjoint, this gives something that can be transformed to the identity. So there should be a natural transformation like this. Since we expect the coevaluation to be equal to the evaluation, this is equal to the dimension of X. So we expect that we'll have a 2-morphism η corresponding to the disk from $dim\ X$ to id_X. There's a natural action of $SO(2)$ here. We saw a circle action, we have a circle action on these morphisms. This η is better than having an element in the space of 2-morphisms. it lives in the fixed points (homotopy) with respect to this circle action. How do they interact with one another? The circle action extends over the disk. This says that the class η is an $SO(2)$ fixed point.

Now let me state a analogue to this two dimensional thing. Giving a tensor functor $Z : 2Bord^{or,noncompact} \to C$ is equivalent to giving a dualizable object in C and a two-morphism $\eta \in 2Hom(dim\ X, 1)^{SO(2)}$ which takes $ev \circ coev$ to id, where η is the counit of an adjunction.

So in $2Bord^{or,noncompact}$ you have objects oriented 0 manifolds, morphisms bordisms, oriented, and two morphisms bordisms with nonempty incoming boundary. For example you allow the disk to the empty set but not the empty set to the circle. This allows us to state the theorem more simply. Without the word noncompact, I'd have to add an additional condition. In string topology you don't have an invariant going in the other direction. If you switch all the arrows around, allow the other disks, you reverse and put things in the opposite order, you get the same sort of thing.

What I'm going to describe will be in the language of algebras. Let me give an example. So I'll let C be an $\infty - 2$ category (over a field k) where the objects are differential graded k-algebras, the morphisms are differential graded bimodules, and composition is an appropriate left derived tensor product. Now I use all maps of bimodules, and then homotopies of maps and so on. This is symmetric monoidal with respect to tensor product with respect to k.

To give a field theory with values in C, I need to give an object and then the 2-morphism. So to any object I can associate its dimension. That's a morphism from the unit object to itself. The dimension is a complex of k-vector spaces. It's the Hochschild chain complex of A but now A is allowed to be a differential graded algebra, either explicitly from Hochschild chains or from our theorem, abstractly.

Say M is a connected closed manifold with basepoint x, oriented, now I can consider a differential graded algebra made out of chains on the based loop space of M, $A = C_*(\Omega M, k)$, and if we compute the dimension you get $C_*(LM, k)$. Moreover, there's a circle action on the homology, from the circle action on the free loop space, from rotating loops.

To promote this to a 2-dimensional field theory. To get η what we want is a map $id_1 \to dim(A)$ which is $SO(2)$-equivariant. Maps from the unit to itself are chain complexes over k. You have a map from k to $C_*(LM, K)$. We have to give a map. That's giving a cycle, but not just any cycle, an $SO(2)$-equivariant one. One way that we can produce such a thing is to start with $SO(2)$ things in the loops, so this receives a map from $C_*(LM^{SO(2)}, k)$ and these are the constant loops. As I've set the dimension up, it should be a 0-chain. You want η to represent the fundamental class of M which lives in the nth cohomology. This η should satisfy something, working it out in this situation, this class η should satisfy Poincaré duality.

I've actually produced string topology for 0 manifolds. To give real string topology I'd need to start over with a twisted version. If I give an appropriately twisted version, I can get arbirtary manifolds.

3 Lurie III

Okay, so let's talk about another example, with C an ordinary 2-category, objects algebras, morphisms bimodules, and 2-morphisms maps of bimodules. This is a symmetric monoidal 2-category. This sits inside the other one, and here my chain complexes live only in one degree. Fully dualizable objects are rare, but there is one object that satisfies it, that's k. The unit with respect to the tensor product is always a fully dualizable object. Then there exists a field theory so that $Z(pt) = k$. This is very boring.

Now you can ask to extend Z to $\bar{Z} : 2Bord^{or} \to C$. What does the theorem say here? Giving this data is equivalent to giving a fully dualizable object of C. This one had the fully dualizable object k. This was supposed to be a fixed point with respect to the $SO(2)$ action. It's trivial on this point. In particular, the unit object in C has the structure of a fixed point, but making something a fixed point is not a condition, it's additional data. How does this spell out in this example? What do you have to do to make this thing a fixed point? You see that you have, well, how should I say this, the first thing that you have to do, is see you have a map $SO(2) \times C^{fd} \to C^{fd}$, and restrict and get a map $\{X\} \times SO(2) \to C^{fd}$ and what you need is a nullhomotopy of this map. This is already the constant loop in C^{fd} taking the value $\{X\}$. So we're trying to give a nullhomotopy from the constant loop. Up to homotopy these are identified with elements in $\pi_2(C^{fd})$. What is C^{fd}? The π_0 are the isomorphism classes of fully dualizable objects. The π_1 are the isomorphism classes of morphisms. The automorphisms of that one-morphism are the automorphism group of k as a k, k bimodule, which is k^*, the multiplicative group of k. There's not much derived stuff going on here. Because this peters out, you don't have to do anything else, but you do need to give an element of k^*. Using the language of the previous lecture, you have the identity map on k and you have S, which is just the identity. Then you need to identify S and k so given $\lambda \in k^*$ you get a field theory $\bar{Z}$ and you can evaluate this sort of thing. Now evaluated on manifolds of high dimension, you can look at what happens on a closed Riemann surface, you get $\bar{Z}(\Sigma) \in k$, and this will be $\lambda^{\pm\chi(\Sigma)}$. How does this square with what you knew before? The only framed manifolds are genus one, so Euler characteristic zero, so if the genus of $\Sigma = 1$ then this number is one.

This is an illustration in a very specific situation and an illustration that being a homotopy fixed point is additional data.

[I have a bone to pick, when I asked if the duality element was extra data, you said no.] That is not additional data, the η which identified the evaluation and coevaluation as adjoints was additional data.

[From the 60s we know that to make a Poincaré duality space a manifold you need to choose Pontrjagin classes. You chose a fundamental class to build this theory, it could have been that there was other data to make other theories.]

Well, there was more freedom in the previous lecture. We needed η to be a map $k \to C_*(LM;k)^{hS^1}$. You could start on the invariant chains, but you could choose many theories here.

[There was an interesting conundrum in the case of the 2-sphere. There's an interesting guess that gave the wrong 2-sphere. The A_∞ structure on the 2-sphere is formal as an A_∞ but not as an A_∞ Frobenius algebra.]

All right, so what should we talk about now? λ can be one. [What does η look like in general?]

That's a smooth proper dgA. It has to be finite as a module over the ground field and a bimodule over itself in the derived sense.

[How do you calculate the saddle disk?]

I stated the theorem before to say, to say what a field theory does you only have to specify what it does on a point and on a disk. If I naturally cut up a two-manifold, I might choose a Morse decomposition, I cut it into handle attachments. The disk I discussed was of index zero. I need to be able to attach disks of index 0, 1, and 2. Before, η was what our field theory did on a disk (empty set to the circle) which gives a map $Z(id) \to Z(ev) \circ Z(coev)$. So η was supposed to be a map from the identity of the unit into the dimension of X. Now η should be the unit of an adjunction, which implies that there is a counit. That tells me that the evaluation map from $X \otimes X^\vee \to 1$ is right adjoint to the coevaluation. This tells me that the other composition is equivalent (by the counit of the adjunction) to the identity on $X \otimes X^\vee$. If I compose the coevaluation with the evaluation I get this picture; the identity corresponds to this one, and this is the picture you get. The compatibility is what you get by gluing a cap onto this with a half-cylinder. This should be equivalent to a half-cylinder.

[Why is the saddle no more extra information?] If I give you two categories, and a functor, I can demand that this functor be a right adjoint. Saying that two things are adjoint is too precise, you should have a reason, the unit is the reason. If you have a pair of adjoint functors and a unit map, that gives you a counit. If you specified both of them seperately, you need to specify some compatibility. That's the sort of thing that I drew here. The saddle is uniquely determined from the way it interacts with the disk. If you like, one way that you can, a more precise statement as to uniqueness is these descriptions of field theories. When you see a basic adjunction, you get a functor from a bordism category, which gives you all kinds of additional data with cancellation andi various compatibilities, which is like the idea that adjuncts are unique up to equivalence.

[If I make coevaluation along with choice of 2-morphisms a category, is it contractible?] Yes, assuming you make all of the demands you should of the situation.

So what do we want to do? [Singularities.] Sounds like fun. I want to go back to the statement of the Baez Dolan Cobordism Hypothesis. This says that $Fun^{\otimes}(nBord^{fr}, C) \cong C^{fd}$, which says that $nBord^{fr}$ is the free symmetric monoidal $\infty - n$ category on one finite dimensional object. What if we want free $\infty - 1$ symmetric monoidal category generated by one object X which is fully dualizable , along with a morphism. What could the morphism be? The objects are disjoint unions of points, so the only things I can consider are tensor products of X with its dual. So I could look at $X \otimes X \otimes X^{\checkmark} \to X \otimes X^{\checkmark}$. I can move things around and put everything on one side. The picture that you might have, here you had five things, three of them were corresponding to X and two of which to $X^{\checkmark}$. Here's a picture of the kind of thing you might want to encounter, so you might want things to be allowed to become singular in some particular way. Here this won't be a manifold but some sort of singular space. Let me, let me define the following invariant of $1Bord$. This is an example of the kind of bordism. For objects you'd want oriented 1 manifolds. Now morphisms are smooth bordisms with singularities like A. I can think of this as a map only among X's. Now the kind of pictures I want to consider, before my one dimensional bordisms looked like this. Now I allow finitely many points where things look like this (valence three picture). This is a picture of the kind of bordism with singularity I want to allow. Before I had diffeomorphisms of bordisms. These are stratified diffeomorphisms. What I mean is, they preserve the singularity structure, It should be a diffeomorphism outside the yellow points and carry those to themselves. The 2-morphisms should be isotopy, and so on.

It might look like this was sort of an ad hoc construction. Let me give an indication that this is in fact very general. The geometry matches up very nicely to a presentation that you might want to consider.

Suppose you want to enlarge $nBord^{fr}$ by adding a k-morphism α which is going to go from $F \to G$, when these guys are two $k-1$-morphisms that we have. See that F and G already live in our universe. Here's a picture of F, and here's a picture of G, they have the same source and target, so the boundaries are diffeomorphic. What is a k-morphism? The ones we already have are the k-manifolds whose boundary are F and G glued along their common boundary. If you're in a situation like this one, you don't need to think of a k-morphism as having a source and target, you think of it as having a boundary. The ability to move things around between the source and the target, that's what dualizability buys you. In summary, if you wanted to enlarge $nBord^{fr}$, you want to adjoin a new manifold which is bounded by a particular manifold that you are given.

Using this data you can define a new variant of $nBord^{fr}$ where we allow singularities that look like a cone $M \times [0,1]/M \times \{1\}$. So $nBord^{fr,C(M)}$ is a symmetric monoidal $\infty - n$ category with objects n-framed 0 manifolds, with the usual definition until k, and then the $k-1$ morphisms defined as usual, then the k-morphisms, the old ones were k-manifolds, now I allow spaces which are stratified so that the open strata look like manifolds, but finitely many points look like cone singularities. The $k+1$ morphisms should be k-manifolds where I have an open stratum that looks like a $k+1$ manifold and a closed stratum that looks like a 1-manifold stuck together via M. The notion of diffeomorphism once we cross n everything is invertible. If we take a limit as $n \to \infty$ and group complete, we get cohomology theories. We have $\infty - n$ categories which model these.

Remark: Many variants are possible. Many things can be considered, such as tangential structures and not only for the ambient manifold but also for the lower dimensional strata. I could also allow many kinds of singularities. Things could become more complicated, later

I could cone off singular spaces. I don't want to consider things that are not iterated cones.

Let me describe the Baez Dolan statement in this setting. Rather than restricting to fully dualizable things, let me assume that C "has duals", meaning that it satisfies strong finiteness conditions. Then $Fun^{\otimes n} Bord^{fr,C(M)}, C)$ is in bijection with objects X in C along with, well, this determines $Z(M)$ and you need a morphism $\alpha : Z(M) \to 1_C$

I realize this was sort of complicated, but I want to give some simple examples where it becomes something concrete. This first example is in the case where $n = 1 = k$ and M was a manifold of dimension $n - k$ and then the kind of pictures I'm thinking about were bordisms which are sometimes like the cone on three points. This is the free bordism category on an object X plus a morphism from $X \otimes X \to X$.

Another example which is fun, let n be arbitrary but let k be one and M be a single point. What kind of singularity do you allow in this point? You have an incoming bordism which hits the singular point and stops. What does it look like when I cone off this singularity? It's a bordism category for manifolds with boundary. What you're saying is that you want to consider not smooth manifolds but the open strata of smooth k-manifolds and closed strata of closed $k-1$ manifold. In this case, the theorem says that giving a functor from this framed bordism category into C is the same as giving an object along with a morphism $X \to 1$. To make this more concrete, let's go back to the kinds of categories we were thinking about earlier. If $n = 2$, we might take C to be an $\infty - 2$ category where objects are differential graded categories, morphisms are functors, and two-morphisms are natural transformations. This is really a generalization of the kinds of two-categories I was talking about earlier. These functors should be linear in a suitable sense. Formulated appropriately is equivalent to giving, well, all functors are given in some setting by tensoring over a bimodule so this is just an enlargement. So a bordism functor is a differential graded category. Now a functor from this particular bordism category is also a functor to the unit, so by representability an object in the category. These sorts of things are naturally encoded by bordisms with singularities.

Remark: These theorems are false if we replace smooth manifolds with PL or topological manifolds. I don't really think I can give a sense of why these things are false without giving details about how these are proved. There's a gap between, if you want, smoothing theory if you have a manifold with boundary, the smoothing on the manifold and on the boundary are a little bit different. To smooth a manifold with boundary you lift the structure groups of the manifold and the manifold with boundary simultaneously and compatibly to different groups.

Anyway, this may be just a technical point, but I thought I should bring it up.

4 Lurie IV

First, does anyone have any questions about anything? All right, why don't I say a few words about definitions that you need to make in order to make sense of these things. The basic objects are symmetric monoidal $\infty - n$ categories. That's sort of a mouthful. I don't want to get all of the details, and so I'm going to define $\infty-1$ categories. I'm going to give a definition that generalizes conveniently as to the n case, and then define the bordism categories in ways that will generalize nicely as well. So $(\infty, 0)$-categories are roughly the same as topological spaces up to homotopy. This is either a definition or take another definition which you should say is reasonable to substitute. It's better for most technical purposes to work with simplicial sets. The idea is that we have a set of objects, for eery object an $\infty-(n-1)$-category of morphisms, and an "associative" composition.

So for an $\infty-1$ category you want for every pair of objects a topological space, so a topological category. It's an ordinary category where on each hom set you have topology and the compositions are continuous.

This is probably the easiest definition, the simplest one to communicate. For many purposes this is difficult to work with. It's hard to establish the right notion of functor for these things. You can always take something associative up to coherent homotopy and straighten out to get things coherent on the nose, but you need to do the same things with functors. If the functor is only compatible up to homotopy, you want to allow these, and it's sort of inconvenient to do this.

Another reason this isn't the definition I want to give, even though every $\infty-1$ category can be so modelled, it's difficult, you have to do a bunch of work. You don't quite get something associative on the nose, it's inconvenient; easier to tweak your definitions.

Another idea. Let C be an $\infty, 1$-category. This is Rezk's theory of complete Segal spaces. I want to imagine I have a definition and extract things. We don't have a definition of an $\infty-1$ category but we do have a definition of an $\infty-0$ category. So we can first extract that, C_0, by discarding noninvertible 1-morphisms. According to this, we can think that C_0 is just a topological space. It's a concrete invariant. It's not a complete invariant because I've discarded noninvertible 1-morphisms. What is a 1-morphism, I can think of that as a functor from $\{0,1\}$, the partially ordered set $[1]$ to C, that's a 1-morphism in C. More systematically we could consider all functors $[1] \to C$. I would expect that $Fun([1], C)$ to be an $\infty-1$ category (as both of its pieces are); Now I can discard the noninvertibles and get a topological space C. The points of C_1 are the morphisms of C. It doesn't know about the composition of noninvertible morphisms.

Now think of diagrams $X \to Y \to Z$ in C. Such a diagram is a functor $[2] \to C$. Then I can consider functors $[2] \to C$ which should be an $\infty-1$ category, I can take and discard the noninvertible morphisms, and I get a space C_2. Now I'm going to write ellipsis, and in general I can consider $Fun([n] \to C)$ as an $\infty-1$ category, discard noninvertible 1-morphisms, and get a space C_n.

Now you might ask, how are these spaces related to one another. Remark: $\{C_n\}$ should form a simplicial topological space, i.e., any time I have an order preserving map $[n] \to [m]$, this should induce a map $C_m \to C_n$.

Now the claim that I'd like to make is that knowing the simplicial space $C_.$ determines the category C up to equivalence. This is not precise without a definition of an $\infty-1$ category.

Let me give a plausibility argument anyway. To recover C, first we need to say what the objects are. These are the points of the space C_0. If I have X, Y, I need to tell you what $Hom_C(X, Y)$ are. I have this map $C_1 \to C_0 \otimes C_0$ which assigns source and target. I have a pullback diagram here. I should form a homotopy pullback here if I wasn't assuming ahead of time that this map was a fibration. A little more surprising is that you can recover composition of morphisms. For composition you can start with X, Y, and Z. So now let's invoke the definition. What is the definition here? I was supposed to take X and Y, cross and get $((X, Y) \times C_1) \times ((Y, Z) \times C_1) \to (X, Z) \times C_1$ These should be homotopy fiber products. Pull these back to $(X, Y, Z) \times_{C_0 \times C_0 \times C_0} C_2$, which then factors through to target $(X, Z) \times C_1$, so you can recover composition with C_2. You want to say that everything is associative and okay up to homotopy. The rest of the C_n say that things are associative up to homotopy.

What's the upshot? We can get from an $\infty - 1$ category a simplicial space, and vice versa. Let's just define an $\infty - 1$ category to be a simplicial space, but let's hold on here, we want the map I've just described to be a homotopy equivalence. A simplicial space is a Segal space if for every n the map $X_n \to X_1 \times X_0 \times \cdots \times X_1$ is a homotopy equivalence.

What is this condition saying? The X_n should parameterize n composable morphisms. A point is a sequence of morphsims. To do that I need only give each of them seperately so that the source of one is the target of the one before it. This is a Segal space, and starting with an $\infty - 1$ category I get a Segal space, so you might think that's the definition. There's a further condition that you want to impose. This condition is just the kind of condition that guarantees that the compositions are well-defined.

You can extract an $\infty - 1$ category by this procedure and get a new Segal space but this might be different than the one that you started with. What is the space C_0? Paths in that space are isomorphisms, which are elements of X_1 which satisfy invertibility. If you want that to recover you need to impose that the only invertible morphisms in X_1 come from paths in X_0. I want to forget this because it won't come up in the most natural bordism category. You can produce one if you need to by first moving to an $\infty - 1$ category and then moving back across.

You want to Hom out of bordism spaces, so if you're going to something complete it doesn't matter.

Instead of talking about $\infty - n$ categories, I want to talk about $(n, n-1)Bord$ as a Segal space.

I need to define X_k, the space of k composable bordisms. I will describe it as a set and you can imagine how to topologise it. If your imagination is precise you may have realized that you wanted to describe it not as a space but a simplicial set. First it will be a sequence of real numbers $t_0, \ldots, t_k$, and second of all it will be M an n-dimensional manifold in $\mathbb{R}^\infty \times [t_0, \ldots, t_k]$. It should meet t_i transversally. This is k composable bordisms. This shows the compositions of all of the bordisms as well. So, um, this varies nicely with k. Suppose I wanted to ignore t_1. I had drawn a picture in X_k; now I've drawn one in X_{k-1}. This is a Segal space; I know how to glue things. Generally I couldn't glue things together because the angles might be wrong. I just need to glue them up to a contractible ambiguity.

So let me remark, in high dimensions this is not a complete Segal space. The invertible morphisms are not diffeomorphisms, they are like h-cobordisms. That would be to say that the invertible cobordisms came from h-cobordisms. That fails in high dimension.

[Shouldn't there be a way to understand the definition where there are no identities?]

You could replace simplicial with semisimplicial. There are problems. You see that there are identities if you say the definition carefully.

[It's nice to imagine that you can replace an identity with a derived notion, and only the notion that $d^2 = 0$.