Focused Research Group Workshop

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November 15, 2008

1 Chris Douglas

[We'd like to start, Chris Douglas will tell about his stuff.] So anyway, I wanted to, Peter and Stephan talked about field theories. They didn't talk much about it but I wanted to focus on the twisting. One of the key ideas, the reason half of us are here is because we're interested in geometric models for cohomology theories, and field theories are good for that

	$* = 0$	$* = n$
H^*	0-d FT	twisted 0-d FT
K^*	1-d FT	twisted 1-d FT
Ell^*	2-d local FT	twisted 2-d local FT

Twisting is not like adding the super, it's like twisting a bundle. I'll start in one dimensional field theories, then I'll talk about twisting them, then talk about two dimensional local field theories, twist those, and then three dimensional local field theories, and then pay back technical debts, about internal higher categories and then in the two dimensional case we'll encounter nets and I'll tell you what those are, and in the end I'll give supporting evidence for why this is a good idea, coming from periodicity and the string group.

Okay, let's start at the beginning, just briefly talking about 1 dimensional field theories. To remind you, it's still early, a one dimensional field theory over X is a symmetric monoidal functor $Bord_0^1(X) \to Vect$ so to each point we get a vector space and to each path a map of vector spaces. Stephan talked about thinking of this as a vector bundle over X (elements in $K^0(X)$. But let's dwell for a minute, maybe we wanted everything to be $\mathbb{Z}_2$-graded, so there's an even part and an odd part, and the K theory element is the difference. and then we can ask, when is this vector bundle 0 in $K^0(X)$? It's if there is an isomorphism between the positive and negative, so an odd operator $e : V \oplus W \to V \oplus W$ and it's of the form $\begin{matrix} 0 & F \\ F^{-1} & 0 \end{matrix}$ such that $e^2 = 1$. So this is the same as saying that there is a Cl_1-action on $V \oplus W$ where this is generated by odd e such that $e^2 = 1$.

In fact we can write this whole K group as vector bundles over X modulo those with a Cl_1 action. Suppose we look at $K^0(X, A)$; now the Cl_1 action is part of the data. An element is a vector bundle V over X together with a Cl_1 action over A, the triviial part.

Suppose we look at $\tilde{K}^{-1}(X)$, which is $\tilde{K}^0(\Sigma X)$ which is $K^0(CX, X)$, so we start to see Cl_1-module bundles over X.

We have Clifford algebras looking like they're related to the other K groups. Let's leave the discussion there and move on to the twisted story.

So now I'll get to answer the question of what exactly I mean by a twisting. Let's talk more generally about what a twisting is. First you have to say what twisting is. Suppose you are trying twist maps $X \to G$. I want a twisted notion. This can be reformulated in terms of the trivial G-bundle over X and talk about sections of that bundle, that's the same as functions $X \to G$. A twisted map $X \to G$ is a section of an arbitrary G-bundle over X.

I will specify τ and say that this is a τ-twisted map:

$$\begin{tikzcd} G \arrow[r] & E \arrow[r] \arrow[d, shift right] & EG \arrow[d] \\ & X \arrow[u, dotted, shift right] \arrow[ur, dotted] \arrow[r, "\tau"'] & BG \end{tikzcd}$$

My model for EG is the paths starting at a fixed point in BG and the projection is the endpoint. So X is sent to $Path_{BG}(*, \tau)$ So x is taken to a path from $*$ to $\tau(x)$.

Now let's think about twisting field theories. A field theory is a functor $Bord_0^1(X) \to Vect$. so I want to twist by $\tau : Bord_0^1(X) \to BVect$ so this will lead to a functor $Bord_0^1(X) \to Path_{BVect}(*, \tau)$.

So BG is a thing so that $\Omega BG \cong G$. So $Vect$ is a category, so $BVect$ is a two category so that $\Omega BVect \cong Vect$. This is just $Hom_{BVect}(*, *)$. What star do you take? You can take $BVect$ to be the two category $CL2$ of Clifford algebras. The objects are Clifford algebras over $\mathbb{C}$, the morphisms are bimodules, and the 2-morphisms are maps of bimodules.

[Is there a categorical notion that justifies the use of the letter B?]

Sort of, this $\Omega BVect \cong Vect$ is a justification. This isn't a connected thing, why didn't I take just one object? It turns out better to build a larger deloop. I think, I needed to have these Clifford algebras, that's one reason to guess this is a good deloop. So now we have a monoidal unit, the trivial Clifford algebra, which is our base point.

Given a $\tau : Bord_0^1 \to CL2$, a two-twist functor, then a τ-twisted one dimensional field theory (over X) is a transformation from the unit functor to τ in $CL2$:

$$\begin{tikzcd} \overline{Bord}_0^1(X) \arrow[r, bend left, "1"] \arrow[r, bend right, "\tau"'] \arrow[r, phantom, "\Downarrow"] & CL2 \end{tikzcd}$$

This is all joint with Bartels and Henriques, motivated by Segal.

We need some specific things, some specific twistings, called degree twistings. So we have Cl_1, and then $Cl_n = Cl_1^{\otimes n}$. There is a functor $-n : \overline{Bord}_0^1(X) \to CL2$ which takes $* \mapsto Cl_n$. Everything here is $\mathbb{Z}_2$-graded, so the tensor product puts in the anticommutativity. My sign for Clifford algebras is that the generators square to $+1$.

The line over the bordisms means that this is a two category, that two morphisms are isomorphisms of one morphisms (one-manifolds)

So now we have the idea of a $(-n)$-twisted one dimensional field theory over X. Take Cl_n as a bimodule for the 1-morphism.

Let's think about what these twisted things look like. We have these transformations

$$\begin{tikzcd} \overline{Bord}_0^1(X) \arrow[r, bend left, "Cl_0"] \arrow[r, bend right, "Cl_n"'] \arrow[r, phantom, "\Downarrow"] & CL2 \end{tikzcd}$$

To a point in x we have a Cl_n-module. So these are related to Cl_n-module bundles over X. Indeed those are related to K-theory, so Karoubi, Wood, and others showed that Cl_n module bundles over X modulo some equivalence gives $K^{-n}(X)$. Now what Peter and Stephan proved is that indeed

Theorem 1 *Stolz-Teichner*
$\{EFT_{1|1}^{-n}(X)\}/\sim = K^{-n}(X)$

[Stephan: you need infinite dimensional Cl_n-bundles]

Let's move on to the two dimensional case. What's a two dimensional nonlocal field theeory? We have $Bord_1^2(X) \to Vect$, but we want it to be a local field theory. This was one of Peter and Stephan's first key observations, that we want this to be a 2-category $Bord_0^2(X)$ and we need, as Stephan says, something larger, and there ar various choices, I like $vN2$; Peter and Stephan seem to have broken up with van Neumann algebras, but we still like them.

[van Neumann algebras are the Banach algebras with a weak topology; that made Alain Connes happy in a different context, so we should be happy here.] A twist should be something like $\tau : \overline{Bord}_0^2(X) \to BvN2$. So maybe I'll just say, if we have such a twist, we can look at

$$\begin{tikzcd} \overline{Bord}_0^2(X) \arrow[r, bend left, "1"] \arrow[r, bend right, "\tau"'] \arrow[r, phantom, "\Downarrow"] & BvN2 \end{tikzcd}$$

and then these will be our twisted two dimensional local field theories.

But what is this deloop $BvN2$? This was our first contribution to this story, which suggested that $BvN2$ should be $CN3$, the 3-category of conformal nets. The objects are conformal nets, the morphisms topological defects, the two-morphisms sectors, and the three-morphisms intertwiners. I'm going to defer what all of these are. But now it makes sense to define

Definition 1 *A τ-twisted 2 dimensional local field theory is a transformation*

$$\begin{tikzcd} \overline{Bord}_0^1(X) \arrow[r, bend left, "1"] \arrow[r, bend right, "\tau"'] \arrow[r, phantom, "\Downarrow"] & CN3 \end{tikzcd}$$

When we build this we'll use all of the [unintelligible]but we don't need anything other than the type one factors. We're in the type one factors.

[Are we in the hyperfinite (injective) world too?] Yes.

In order to keep going, we need a notion of degree twists. Oh, I forgot something. This is what happens when you prepare the night before.

[As opposed to the morning?] Well, more like the week before. Okay, let's go back, the basic idea behind the two dimensional local field theories, if you take these over X and mod out by equivalence relations, do you get $TMF^0(X)$? Implicit is a bunch of stuff, smoothness and supersymmetry and so on.

Now we want to contact TMF. It is possible to make something precise for $LFT_2(X)$, but there are lots of different things to write here and it's not yet clear which one will be TMF. Now what are the degree twists? We have these Clifford algebras. The first thing is the analogues to Cl_1 and Cl_n. There exists a conformal net $Fer_1 \in CN3$, called "the free fermion" and we can define n free fermions Fer_n to be $Fer_1^{\otimes n}$, and then a calculation shows that there exists a functor $\overline{Bord}_0^2(X) \to CN3$ which takes $* \to Fer_n$, and this can be called, again, $-n$. This now exists, and now I can talk about a $-n$-twisted two dimensional local field theory, and now I can ask the following question. If I take $-n$-twisted local field theories over X and mod out by an appropriate equivalence relation, do I get $TMF^{-n}(X)$? That's the moral story.

Okay, so maybe before, well, just to summarize, we have the notion of field theory that we're looking at and the cohomology theory we were hoping to get

	FT	cohomology theory
one dimensional	$Bord_0^1 \to Vect$	K^0
twisted one dimensional	$\begin{tikzcd} \overline{Bord}_0^1(X) \arrow[r, bend left, "Cl_0"] \arrow[r, bend right, "Cl_n"'] \arrow[r, phantom, "\Downarrow"] & CL2 \end{tikzcd}$	K^n
two dimensional	$Bord_0^2 \to vN2$	TMF^0 (we hope)
twisted two dimensional	$\begin{tikzcd} \overline{Bord}_0^1(X) \arrow[r, bend left, "Fer_0"] \arrow[r, bend right, "Fer_n"'] \arrow[r, phantom, "\Downarrow"] & CL2 \end{tikzcd}$	TMF^n (we hope)

[What is the assumption on n in the theorem of Koroubi?] There it might be n positive.

[Without Bott's theorem, you wouldn't know that this is a cohomology theory yet.]

But it should be easy to see periodicity here because of the Clifford yoga. Once you know you're getting K-theory you can use Clifford yoga to see periodicity.

Now let me talk a bit about 3-dimensional local field theories. Some people use the word extended, some people omit it entirely, I like the word local.

[In the absence of a theorem, is it possible that the left hand side is just empty? Are there examples?] Yes.

[For those of here who haven't been toiling a lot with the extended idea, it was alluded to by Mike Hopkins and referred to by Stephan. Local corresponds to point, interval, et cetera. It hasn't been brought in as evidence.]

Roughly speaking, $Bord_0^3$ is a three category whose objects and n-morphisms are n-manifolds (with corners). So a three dimensional local field theory is a functor $Bord_0^3 \to CN3$. The philosophy here is that because we've gone local, everything should be determined by what happens on points.

[Are you talking topological field theories now?] Yes.

[Why is this notion good for locality?] You can take, well, what is the initial idea? You can cut your manifold into pieces, cut it up, and now I can chop that in a different direction and get smaller pieces, and whatever I'm left with, by construction, I can build up more and more complicated things just with the data. That brings up a natural question, which values (for the point) are okay.

So let me bring back up the Baez Dolan hypothesis, which was mentioned yesterday, local field theories with values in C^d should be in correspondence with fully dualizable objects in $\mathbb{C}$.

Let me write out cases by way of advertisement. So I'm going to focus on two cases. One is the framed case in dimensions two and three, and then the topological case in dimension two and three. We worked out the first three, BDH, and Hopkins, Lurie had also done that. Now Schummer Pries had done the $n = 2$ topological case and he and I did the $n = 3$ case, and now it looks like Hopkins-Lurie have done all n, Jacob will talk about that.

So $n = 2$, a 2-dualizable object, I'm going to do something confusing. They have a target C^2. I'll call a target in C^2 by its coimage in $Bord_0^n$. So this is an object $\bullet$ so that there exists another thing $\bullet^-$ so that there exist one morphisms $\subset$ and $\supset$ from $\bullet \sqcup \bullet^-$ to the empty set and vice versa. Then there are two morphisms, cap, cup, up saddle, and down saddle, which have a relation I can draw geometrically. This corresponds to the $F \to FGF \to F$ we saw yesterday.

I'm going to abbreviate the points as m and n, I'll project the 1-morphisms and look at the singular points, projected to one fewer dimension. So the cup, cap, and saddles all look like $\cup$ and $\cap$. Then the relation smooths something looking like a cubic to a line.

So let me tell you the three dimensional analog. So a three dualizable object is m so that there exists n, a one morphism $\bullet$, the caps and cups, morphisms $\bigcirc \to \emptyset, \asymp \to \|$, and then something that kills a bump.

So I have relations. If I create and kill a bump, that's the identity on the line, if I kill and create a bump, that's the identity on a bump. If I move from a cup to a circle with $\| \to \asymp$ and then kill the circle, that's the identity on $\cup$.

There are two more relations. If I start with a cup, introduce a bump on the left, and then on the right kill that bump, that's the identity, and if I start with a bump over a cap, switch it to a line and a cap, and then switch back to get the opposite bump over the opposite cap, that's the same as doing it in the other order. The fourth one here is called the swallowtail relation, and you will see it again in Jacob's case.

I'll come back a little later and give at least one example of a field theory. Now I'll try to start repaying my debts. Dennis has asked what all these things mean. I want to talk a little bit about internal higher categories. Let's start at the beginning with 1-categories. A 1-category has objects and one-morphisms, which together form a strict one category. If we didn't know better, we might have done this vertically and thought of them as zero and one cells and thought of it as a weak one-category, but that's the same. Let's go to two-categories. It has objects, one morphisms, and two-morphisms. I could also look at things like two cells. One-cells compose weakly, up to two-cells. So that's called a bicategory. I've drawn it to suggest that there's something in the middle. I can position vertically and say that what's a category in Cat? C_0 is a category of objects, C_1 a category of 1-cells. Then there's a composition functor, an identity one-cell, and then a bunch of ways for these to be related, like if I compose in the two orders, there's a natural transformation, and there are identity transformations. This is all the data of a category object in categories, and it satisfies a bunch of axioms. Using this notation it's fast to see the axioms. One comes from composing two identities, three from composing an identity with two other things, and then one for composing four morphisms, which is the pentagon.

Well, I'll keep going, what happens in the case of three-categories? We can draw the same picture. We have 3-categories (0-categories in them), weak 3-categories (tricategories in sets) and then in between categories in $2Cat$ and bicategories in Cat. A category in $2Cat$ has objects and one-cells which are 2-categories.

An example of a category in Cat is algebras. So we have algebras, maps of algebras, bimodules and maps of bimodules. That's a category in Cat.

So examples, $\mathbb{C}$, $Vect$, $2Vect$, and $3Vect$. You have functors of modules over the bicategory of vector spaces. Let me fill things in. To go down, there are good reasons to write $B\mathbb{C}$ as a deloop of $\mathbb{C}$, we can deloop many times. I told you that Alg is a good category object in categories. Tensor categories are a good category object in two-categories:

$$\begin{array}{cccc}
\mathbb{C} & Vect & 2Vect & 3Vect \\
B\mathbb{C} & Alg & TensCat & \\
B^2\mathbb{C} & Nets & & \\
B^3\mathbb{C} & & &
\end{array}$$

The punchline is that this gives you a very nice way to deal with the symmetric monoidal structure. I can just work in a bicategory in $SymMonCat$ or something.

Let's go to nets, let me describe the Fermion. If I have the interval, I can look at $L^2(I)$ and then form a Clifford algebra on $L^2(I)$. This is functorial. So you see that this way of taking an interval and producing an algebra is Fer which gives motivation for the following definition. A net is a cosheaf of von Neumann algebras on the category of intervals (with maps being inclusions. You extend a function on an included interval by zero) A precosheaf is a covariant functor. cosheaf means it satisfies gluing properties. What are the six things?

For nets, this is

nets	maps of nets
defects	maps of defects
representations (sectors)	maps of sectors

Maybe in the last few minutes I should say a word about evidence. Let me talk about periodicity. In one dimension, $K^{-n}(X)$ was Cl_n module bundles up to equivalence. It only depends on Morita equivalence classes of Cl_n and then Atiyah Bott Shapiro proved that $Cl_8 \cong Cl_0$ so as a corollary is that K-theory is 8-periodic (real). In the complex case it's true with period two. What's the idea in the two dimensional case? We have the two category of conformal nets. So now we have for the purpose of elliptic cohomology the moduli space of tori, and maps from that into $CN3$. The homotopy of that mapping space is controlled by the homotopy of $CN3$ and the cohomology of this moduli space. In my fantasy land, the question of the periodicity of the free Fermion has to do with $\pi_0(CN3)$. So we could have $\mathbb{Z}_{24}$ and we might see $\mathbb{Z}_{12}$ and $\mathbb{Z}_2$ in $H^*\mathscr{M}_{tori}$, and they could have a party and maybe in the morning we might see $\mathbb{Z}_{576}$, the periodicity of TMF. I should have said, twenty-four suddenly appeared. Here's what we can prove. If $Fer_n \cong Fer_0$, then $24|n$, so this was the first thing we figured out. Conformal nets weren't the first thing we tried, we started with tensor categories. We never found a 24, so we've only seen that in conformal nets. That's periodicity.

I'm over time, but let me say we can produce the string group from $CN3$ in a canonical way, which is closely related to TMF, which is more good data. If you look at, well,

Theorem 2 *(DH)*
$Hom_{CN3}(1_{Fer_n}, O(n)) \cong String_n$

2 Kevin Costello

Thanks, Dennis, I suppose what I want to talk about is how we could have discovered Feynmann graphs and so on by thinking about rational homotopy. This should be easier and more relaxed than my talk yesterday. I'm going to start off with something everybody knows. If M is a compact manifold, oriented, we'll spend a lot of time considering some quantum field theory but before that we'll start simply and note that $\Omega^*(M)$ is a differential graded algebra and according to Dennis' work in the 70s, Quillen's work, this encodes a huge amount about the manifold M, the real homotopy type. What I'm going to do is start off by showing how you can show something using the harmonic forms, which are finite dimensional so much more practical.

Let's pick a metric on M. We have operators d, the de Rham differential, and $d*$, its adjoint, and the Laplacian $\Delta = [d, d*]$. From the Laplacian we can see the space of harmonic forms, $\mathscr{H} = Ker\Delta \subset \Omega^*(M)$ which is the same as the homology.

We want a way to encode this differential graded algebra. So $\pi : \Omega * (M) \to \mathscr{H} = H^*(M)$, so $\mathscr{H}$ is an algebra. The algebra structure is $h_1 * h_2 = \pi(\iota h_1 \wedge \iota h_2)$. We see a problem immediately, which is that the inclusion is not an algebra map. If we take the product of two harmonic forms, it's not harmonic. You have to project back down, this map is not an algebra map.

Okay, so, does anybody have the eraser? [With that definition of $*$, does π become an algebra homomorphism?] No, you can take two things in the kernel, and you might end up with harmonic pieces.

Okay, the problem, the cohomology is not enough to encode everything, as a commutative ring. What we'd like to define is some better structure on $\mathscr{H}$ which does encode everything. We're going to vary the algebra structure on forms by a homotopy parameterized by $[0, \infty]$. At 0 we'll have the original structure; at ∞ they will be a subalgebra, but the price we have to pay for being able to do this is that the algebra structures we get will be A_∞ structures.

So I'm going to define $m_2^t : \Omega^*(M) \otimes \Omega^*(M) \to \Omega^*(M)$ so $m_2^t(\alpha, \beta) = e^{-t\Delta}(\alpha \wedge \beta)$, where forms are completed to L^2, and this $e^{-t\Delta}$ preserves the smooth structure.

This makes things smoother and smoother and eventually harmonic. Now $m_2^0(\alpha, \beta) = \alpha \wedge \beta$ and if $h_1, h_2 \in \mathscr{H} \subset \Omega^*(M)$, this is $m_2^\infty(\alpha, \beta) = e^{-\infty\Delta}(\alpha \wedge \beta) \in \mathscr{H}$.

There's an obvious problem with this which is that it's nonassociative.

We can take α and β, let them collide, do the wedge product and then the smoothing operator. Mike Douglas was saying that functions are like particles, let them collide, and then evolve by time. Why is it not associative? I'll draw the diagram for $m_2^t(\alpha, m_2^t(\beta, \gamma))$. We take β and γ, multiply them, and then evolve them for some time, and then hit that with α and then evolve for some more time. Do you agree? On the other hand, what is $m_2^t(m_2^t(\alpha, \beta), \gamma)$? We draw the same diagram but things are flipped around a little bit. First α and β meet, then they evolve and meet with γ and then evolve, and these two things are just not the same. This is where we find this clever trick of saying, it's okay if it's not strictly associative, as long as it is associative with respect to a specified homotopy.

We need to find a specific homotopy $m_3^t : \Omega^*(M)^{\otimes 3} \to \Omega^*(M)$. We want this to be a chain homotopy, and what we need is that when we commute this with the differential, and then apply to α, β, γ, this should be the associator of the two ways of using m_2^t to combine the three arguments.

Okay, so, at least this is the first start, we'll also need higher and higher homotopies. We can produce m_3^t. We should think, if we could take the long edge, squish it down, and pull it out in the opposite direction, that would give the desired answer, that's exactly what we're going to do.

We're going to use this, can I erase this stuff? So to construct the homotopy we need something for a family of trees. Let's look at this integral

$$S^t = \int_0^t d * e^{-\tau\Delta} d\tau$$

well, $[d, S^t]$ is, d commutes with everything except $d*$ so this is $\int_0^t [d, d*]e^{-\tau\Delta}d\tau = \int_0^t \Delta e^{-\tau\Delta}d\tau = 1 - e^{-t\Delta}$

Most of these pieces commute with the differential. Everything commutes with the differential except S_t. Maybe by clever diagram reuse, I can say what's going on. If we commute this with m_3, we replace the homotopy with $1 - e^{-t\Delta}$. So now the commutator of m_3^t with d is the difference between the two ways of bracketing.

What we really want, if you read Stasheff's thesis, having m_3 is not really enough, you want to be able to stick m_2s in the middle of it. What we really need is a homotopy associative structure, you need maps $m_n : \Omega^*(M)^{\otimes n} \to \Omega^*(M)$ of degree $2 - n$ so that, there's some funny identity, and a very good reason for this identity. When we commute this m_n with the differential the same way as before, we find some stuff.

$$[d, m_n^t] = \sum_{r+s=n+1} \pm m_r^t(\cdots m_s^t(\cdots)\cdots)$$

What is m_n^t? We just mimic what we did for m_n^3. It's the sum over all trivalent rooted trees (with sign) of, every time you see an internal edge you put a homotopy. The external edges have fixed length, the incoming ones are decorated by αs, and then the root is decorated with $e^{-t\Delta}$. This is particles moving some distance and then moving proper time, and so on. When $t = \infty$ it's the same formula, well, you find that it preserves the subspace of harmonic forms. It's the A_∞ structure on harmonic forms. You can write this structure as a functional integral.

Let's figure out what happens when this commutes with the differential? When we commute it, we find that everything respects the differential except the homotopies. So precisely one homotopy is replaced with its differential. So we replace that one homotopy with $1 - e^{-t\Delta}$. This $e^{-t\Delta}$ edge divides the tree into two pieces. So the boundary is a sum of the lower pieces.

I want to explain, there's a version of this having to do with the Feynmann graph expansion, only the classical part.

Let me make the functional integral, and then write a theorem and then maybe a break. Take $\mathfrak{g}$ a Lie algebra with an invariant pairing, and look at $\Omega^*(M) \otimes \mathfrak{g}$, a Lie algebra. Then we find that $\mathscr{H} \otimes \mathfrak{g}$ is a homotopy Lie algebra structure. This is given by some maps exactly as before l_n, we can write it on harmonic forms or on, well, $l_n^t : (\Omega^*(M) \otimes \mathfrak{g})^{\otimes n} \to \Omega^*(M) \otimes \mathfrak{g}$ The l_n^t are given by the same formula with Lie brackets instead of the product. I'd like to give you a functional integral.

[When you say that $\Omega^*(M) \otimes \mathfrak{g}$ is a Lie algebra, it's not skew symmetric?] It's graded skew symmetric.

So the formula is as follows. Let's fix an $\alpha \in \Omega^*(M) \otimes \mathfrak{g}$ and let's look at the following expression: $\sum \frac{1}{n!} \langle l_n^\infty(\alpha, \ldots, \alpha), \alpha \rangle$, which is like the Taylor expansion of something on a vector space, and you can write this as (this is the classical Chern Simons functional integral

$$\lim_{\hbar \to 0} \hbar \log \left(\int_{\phi \in \Im \ d*\otimes \mathfrak{g}} exp(\frac{1}{\hbar}(\langle \phi, d\phi \rangle + \langle [\phi + \alpha, \phi + \alpha], \phi + \alpha \rangle) \right)$$

So we're taking the limit as $\hbar \to 0$, which corresponds to selecting the graphs which are trees, those that define the homotopy structure. Then there's the rest of the expansion, what happens if we don't take this limit.

Theorem 3 *One can renormalize this integral without taking $\hbar \to 0$ and the result is an algebraic structure, we find some quantization of the rational homotopy type, or of the usual ∞ structure on $H^*(M) \otimes \mathfrak{g}$*

So the simplest version of this would be $dl_2(\alpha, \beta) = \sum \hbar l_4(e_i, e_i, \alpha, \beta)$

3 Kevin Costello III

A lot of the things Mike and Peter and Stephan have talked about have been Segal axioms for field theories in terms of higher categories and functors. In my point of view, can one see anything like the Segal axioms. I'm going to start by talking about observables.

Imagine you're a physicist. You have U in some spacetime manifold M and some quantum field theory on spacetime. [What is that?] Who knows? You might imagine that whatever a quantum field theory is, you can measure things, we can take our detector, put it in U, and measure what happens in U. So there $Obs(U)$ which is a vector space of measurements we can make on U. In a minute I'll give it a more precise definition, but for now it's reasonable to postulate that there might be such a vector space. Another way to think is that this should be something like, some quantization of functions of the field φ which only depend on φ's behavior on U. φ might be a function on M or a section of a bundle. We should have to do something clever to our function to turn it into a quantum observable. In a minute I'm going to explain, well, in terms of yesterday's lecture, let's work with scalar field theories again. Suppose, after yesterday, everything was determined by low energy effective actions $\{S^{eff}[\Lambda]\}$ is a system of effective actions, defining a theory, so then you can ask how do you see what the space of observables is in these terms. So observables are like first order deformations, the tangent space to the moduli space of theories. Define the space of observables of U to be the space of first order deformations $\{S^{eff}[\Lambda] + \delta O[\Lambda]\}$ So δ is a parameter, you can work modulo δ^2. This is like, well, it will satisfy the renormalization group equation modulo δ^2, the quantum master equation modulo δ^2, and there's one more axiom. Where's my eraser?

So you remember when we defined quantum field theory, these had to satisfy a locality axiom, and O should satisfy an axiom as well, as Λ goes to ∞, $O[\Lambda]$ becomes supported on U. This is the part that makes it not the tangent space.

Everybody reasonably happy with the definition of observable? Why should we imagine observables are first order deformations? Suppose we start with S and do a first order deformation. Then if we try to integrate we get

$$\int_\phi e^{(S+\delta O)/\hbar} = \int_\phi e^{S/\hbar} + \delta \int_\phi e^{S/\hbar} O + O(\delta^2)$$

So this is great, what properties would we expect observables to have, what do they have?

So firstly, if $U \subset V$ then there's a map $Obs(U) \to Obs(V)$. This is because the third axiom is weaker when V is an open set. This means that $Obs(U)$ form a precosheaf, which is a completely horrible word. They're only vector spaces because, if you wanted to multiply them, $O[\Lambda]O'[\Lambda]$, well, the renormalization group equation is not preserved.

We saw this same thing in Mike Douglas' talk. If U_1 and U_2 are disjoint, there's an isomorphism between observables under union and observables under tensor product, $Obs(U_1) \otimes Obs(U_2) \cong Obs(U_1 \cup U_2)$.

I suppose I want to say, these axioms are not really sufficient. We have no restrictions. In fact, they satisfy a stronger and slightly technical set of axioms of a factorization algebra. I'll just tease you with this. What can we do with this structure? Let me give a name to it. Something satisfying these two axioms is a net of vector spaces. So what about nets?

What is a net on $\mathbb{R}$? It's not so different from an associative algebra.

In QM we have a net, observables on $\mathbb{R}$. So $Obs(a, b)$ is a vector space for all $a < b$ and a map $(a, b) \to Obs(c, d)$ if $c \leq a < b \leq d$, and a map $Obs(a, b) \otimes Obs(e, f) \to Obs(a, f)$ if $a < b < e < f$

This is a strict generalization of an associative algebra. Suppose we have some kind of net on the real line where the inclusion is an isomorphism. Then all of our vector spaces are canonically isomorphic. So we have only one vector space, which is isomorphic to $Obs(a, b)$ for any a, b. This is an associative algebra, because we get a bilinear map. It's associative because of the axiom I forgot to write down. We know what happens when we have two open sets that are disjoint. We want all of the isomorphisms when we have three disjoint sets to commute, so it's associative.

Now what if these are quasiisomorphisms. We don't have a product on the vector space, but on something quasiisomorphic. We can turn this into a homotopy associative product.

[How do you get a differential on observables?] This is something you should know. It comes from the BV master equation. We take our observable. The fancy way to say it, is we can look at homotopies of solutions, so make a simplicial set, and then observables are a simplicial Abelian group. Then this is respected by the renormalization group equation.

[Your observables are classical observables?] No.

[How is this related to the naive quantum mechanics?] There's something strange happening here. The inclusion maps are quasiisomorphisms there. Let's work with quantum mechanics. Use the quantum mechanics of maps on the real line. So $\phi : \mathbb{R} \to V$ a vector space with inner product. This is a free theory so the action is just going to be $\int_{\mathbb{R}} \phi(\frac{\partial}{\partial t})^2\phi$ and when we do this construction with BV formalism we find that inclusions are quasiisomorphisms I'll write down the observables on the whole real line

$$Obs(\mathbb{R}) = \prod_{n\geq 0} Hom((C^\infty(\mathbb{R}, V) \xrightarrow{D} C^\infty(\mathbb{R}, V))^{\otimes n}, \mathbb{R})^{S_n}$$

The observables at a point (the inverse limit of open sets containing that point) $Obs(x) = Sym^*(\mathbb{C}[\frac{d}{dt}] \xrightarrow{D} \mathbb{C}[\frac{d}{dt}])$. Only in dimension one, this has finite dimensional cohomology.

The point of this, because of the quasiisomorphism property, we should get an algebra, which algebra is it? It's an algebra of observables, the Weyl algebra on $(V \oplus V^*)$ generated by V and V^* modulo $[V, W] = \langle V, W\rangle\hbar$. The inner product has given an isomorphism between V and V^*.

[What was the master equation?] [unintelligible]. This is an associative algebra, the higher products vanish.

What happens if we work with quantum theory for a manifold?

Let's let M be a manifold with boundary N and let's just like as in Peter and Stephan's talk choose a collar around the boundary. Then what we find is that to the boundary we associate a net on the real line, so something like an algebra. The observables restrict to $N \times [0, 1) \to [0, 1)$ via p and then we can define a net p_*Obs which gives $U \mapsto Obs(p^{-1}(U))$. In this simple way we see that we have associated to the boundary something that looks like an algebra. Let's call this net A_N.

Now this is a reasonable thing to do?

For quantum mechanics we have the algebra of observables, now we have the net of observables. Let me make a definition for the space of states. These are supposed to be the initial conditions of your field. If M has boundary N then a state is a module for the algebra associated to N, then an element in this module, probably nonzero. We see the usual states are elements of a module over $W(V \oplus V^*)$

Firstly states are in a projective Hilbert space and secondly they can be added. Direct sum is superposition. In quantum mechanics there is a unique irreducible module. In quantum field theory there isn't. The good states might be irreducible modules. In the usual formulation of quantum mechanics the Weyl algebra you find is semisimple and has a unique irreducible module.

In quantum field theory, there's many modules $\mathscr{A}_N$ modules are in correspondence with boundary conditions. If we consider a scalar field theory, maybe one other philosophical point. Here we've associated to the boundary something like an algebra. The general philosophy is that the space of, a quantum field theory with an action, the space of classical fields solving the equation of motion defined near the boundary, this space is always symplectic. $\mathscr{A}_N$ is like a deformation quantization. Now we can see why there is more than one module.

So for example take the free field theory on M with boundary N and action $S(\phi) = \int \phi\Delta\phi$. So germs, the equation of motion is that ϕ is harmonic, and germs of solutions to equations of motion on N will be two copies of the space of functions on N, so $C^\infty(N) \oplus C^\infty(N)$. One is evaluating the field on the boundary, the other its inward normal derivative. One boundary condition (there are two, Dirichlet and von Neumann), is that the $\phi|_N = 0$ and the other that the inward pointing normal is zero, corresponding to the two Lagrangians of the symplectic vector space.

[In this example, will $\mathscr{A}_n$ always be an algebra with $\hbar$?] [unintelligible]

These two modules will give two modules which are not isomorphic, and neither is, say, a quotient of the other. They are separated by an infinite amount of energy. These two Lagrangians cannot be homotoped to one another.

I'm getting a little incoherent. Maybe I should stop soon. Maybe I should just spend five more minutes explaining what the manifolds do and then we could stop?

[Is there any sense in which [unintelligible]relates to the dualities you hear about?] I don't think so.

Another thing that comes up, here we have N_1 and here N_2 and some M in the middle, now we have these A_{N_1}, a net on the interval, and A_{N_2} a net on the interval going the other way, and in the middle a bimodule for these guys. The space of observables for M is an A_{N_1}, A_{N_2} bimodule with an element. The observables are a functor between the states on N_1 and N_2.

Because N_1 has a map $N_1 \times [0,1) \hookrightarrow M \hookleftarrow N_2 \times (2,3]$, we have maps $\mathscr{A}_{N_1} \to Obs(M) \leftarrow \mathscr{A}_{N_2}$. There is a canonical observable $1 \in Obs(M)$ always present. Take the identity in the two algebras and then map it by inclusion and that's the state.

We find that, e.g., Yang Mills theory from, say, Euclidean bordisms $Bord_3^4$ to categories, this is a twisted theory. You have another functor of boundary conditions, and there is a natural transformation between these two fibered by vector spaces.

$$\begin{tikzcd} Bord_3^4 \arrow[r, bend left, "\text{states}"] \arrow[r, bend right, "\text{boundary conditions}"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Cat} \end{tikzcd}$$

[What does it mean to be a module over a net?] You have something for every interval containing zero, and you can multiply on the right, like that. [Picture]