Focused Research Group Workshop

Gabriel C. Drummond-Cole

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1 Hopkins

[Our first talk today is by Michael Hopkins, he'll tell you what the title is.]

Let's just, we can make up the title at the end.

So a few years ago, when my daughter was in kindergarten, this meeting went on forever. This guy next to me said that we have a saying in the Marines, they crammed a five minute meeting into an hour and a half. I thought I'd do something like that but now two hours seems small.

Everything here is joint with Jacob Lurie. The results are in some sense a classification of topological field theories in all dimensions. Our formalization and conceptualization is heavily influenced by Baez and Dolan. They make various hypotheses about manifolds. Our work is a mild refinement of their work and then some proofs. We heard yesterday some general things about field theories in general. Let me talk, before trying to give a definition and a classification, about what we need theories to be. I'm going to talk about an n-dimensional topological field theory.

We've heard that a field theory is supposed to associate to manifolds $\overset{Z}{\rightsquigarrow}$ some other mathematical object. This other object might be a vector space, linear category, algebra, this depends on the dimension and structure of the manifold. For now let's stay general. This should express two things. One is locality and the other is superposition. Both of these came up quite a bit yesterday. Locality has to do, these manifolds are like space-time, and locality tells us that we can calculate evolution of this theory by calculating with small intervals with time and with space.

Let me try to be a little more formal. To every zero manifold, we'll get a mathematical object. To every one-manifold with boundary we'll get a mathematical object.

I might have a two-manifold with corners or something, and I'll need to associate data to this, et cetera. I need to eventually get to the dimension of space-time, so this will stop at n. The kinds of field theories we've heard presented, you take space-time or a manifold, and cut it along codimension one pieces, and imagine looking through these kinds of cuts; I'd like to imagine cutting it smaller.

There are a couple of things I want to abstract from this. Locality happens when I glue this together, gluing along these interfaces. Superposition has to do with some behaviour of Z under disjoint union.

As mathematicians, I want these to move in families, so I want to consider all of these kinds of manifolds glued together and I want them to move in families. So I want to take bundles of these over parameter spaces.

I'm not going to give a real careful definition of this structure, I want to get to the terms of this classification. That journey gives better information about the kind of structure than giving a careful definition.

We already know that the language of categories captures some of this structure. We might imagine that to each zero manifold we get an object, to every bordism a map of objects, and our gluing gives us a composition, and something like a two-manifold will be like a map between maps, and as we continue, we're looking at a 3-category, an n-category, and part of the n-category has to work like spaces. When it's combinatorial we use a combinatorial name, like "category," and when you introduce the family the adjective that this introduces is ∞. So we want to express these things by saying that we have some kind of $\infty - n$-category of manifolds. Let me just state something. The topological field theory will be a functor to $\mathscr{C}$, an $\infty - n$-category.

That structure captures the locality. That's only two and a half minutes. I also have to capture superposition, and so in this case I have the disjoint union of manifolds, and on the other side I need to have an operation like addition, so like a $\otimes$ (symmetric monoidal structure). An n-dimensional TFT will be a functor and compatible with the symmetric monoidal structure.

Classical theory says that when I have two events, the partition functions add. I'm using the category language for a generic commutative combination of things. I want to classify these. One of the nice things about having these structures, running through this focusses the discussion and highlights the important features, the things that would be a good definition. I want to go through here without a careful definition.

Now, so let me try to get to Dennis' question about the ∞ thing. I want to give an example that we can refer to a lot. So $n = 0$ or an $\infty - 0$ category. All I have is a set, but I should allow my elements to move in families. So this should be a topological space. That's great but I want to think about this in the language of categories.

If I have X I can make a category, where the objects are the points, the maps are the paths, a 2-morphism is a disk. This is an ∞-groupoid. Why is the adjective ∞? Because all of the compositions are not associative but infinitely coherently associative. The equivalence of such categories is weak homotopy equivalence.

I think the attitude to take is to learn to do algebraic topology with $\infty - n$ categories. There's an old trick due to John Moore, if you allow the space of pairs, a path and a t, then you can make composition strictly associative. You've slightly inflated the path space but you haven't changed the type. You can increase the size of the 2-morphisms, and you can do this all the way. You can always rectify the ∞ homotopy at the cost of increasing the size of the space a bit. I want to say that there's a very clean way to talk about $\infty - n$ categories. There's a good notion of a strict n-category. You can push all of this up to the highest level and say that an $\infty - 0$ category is a space, an $\infty - n$ category is a category with morphisms $\infty n - 1$ categories. Any other model is an implementation or equivalent to that one. These aren't that crazy of notions and there are always nice models.

I'll get to $\infty - n$ categories in a minute.

[I was expecting you to write down covering spaces or $n : 1$ maps.] When I say I am putting ∞ in, I'm allowing it to move in families. If I allow families of discrete sets, well, if I take an ∞ version of a point, it's a space. I'm passing from the category of discrete sets to spaces.

If I'm going to, well, I can think of a space as a category, or I could think of things moving in families.

[There's an equivalence class, and he doesn't want to give details because the details are dependent on choices.]

There's this famous trouble in formulating a three-category. You can't form the Whitehead product, if you allow your three-morphisms to form a space instead of a discrete set. I found that sort of convincing. I agree that I'm suggesting there's one theory you can write down. I'm suggesting that this is the right idea but particular implementations are in its equivalence class.

I want to borrow something from this picture. An $\infty - n$ category is an ∞-category but with all m-morphisms equivalences for $m > n$. Every path can be drawn in the opposite direction, and the composite is homotopic to the identity, et cetera. If C is an $\infty - n$-category, then it's also an $\infty - m$-category for $m \geq n$. Here ∞-category is an $\infty - \infty$-category. In my example, I could have said that topological spaces are an $\infty - 3$-category, but everything starting from 1 is invertible. I want to say that structure goes all the way to ∞, but beyond n all of the structure becomes invertible.

There's another piece of structure that I want to motivate.

[Can you show why the two-morphisms are invertible?] Suppose here's two paths, I can take the disk spanning them, and then the opposite disk, those bound a ball.

[Technical category question]

One of the reason you can get away with an $\infty - 1$ category as a simplicial thing is that you can say when a one-cell has an inverse. If you're going to model it on simplicial sets you need to add structure. There are versions built on multisimplicial sets. To model two-categories you need extra structure to model the invertibles.

[Good things for a parameter space should be functors of parameter spaces. Is that true for

$\infty - n$-categories?] I think so.

What about the symmetric monoidal. That's a space with an infinitely coherently commutative binary operation. That was studied by topologists in the 60s or 70s. That's a spectrum or an ∞ loop space or a Γ-space. We know how to do this. So when $n = 0$ this is under control.

i want to combine these to say something important. Let's say I have an n-TFT, so a functor $Bord_n \to C$, and suppose C happens to be an $\infty - 0$ category. It might so happen that every morphism is invertible. This reduces to something in algebraic topology. So this then reduces to studying a map of Thom spectra to another spectrum. Later in my talk I'll work out and explain what it is. This is a combination of old stuff with Galatius [unintelligible]Tillman Weiss. So in this case we're on home ground and classical techniques tell us that we can solve this. So we're calculating a space or spectrum of maps.

[When I first went to France in 1974 Thom said that categories weren't right, they needed an ϵ and to have a little more room to move things around more]

Is that right?

Okay, in a TFT things aren't all invertible but there is some structure left to play with. Suppose I have a transformation represented by a cobordism, here's one from 0 to two points. I can read this backwards. It's not that, well, is that the inverse? I can go the other way. Let me draw these with a little bit of perspective. [pictures] If I want to relate the composite to the identity map, I can do that but this introduces a topology space, so it's not necessarily invertible. We do have, given a map, there's always a map in the other direction and then a relation between the composition and the identity, but they're not always invertible.

What this should remind you of is the notion of adjoint functors. If I have, we have to go up a category level. Suppose C and D are two categories, and we have this notion of adjoint functors F and G, $Map_D(Fx, y) = Map_C(x, Gy)$, so you know that G is determined by F, it's unique up to unique natural isomorphism. Now, this is the usual formulation, but to say it this way goes against the philosophy of category theory. I should have expressed that in terms of mapping properties in Cat. This is expressed as, there should be a map $\mathbf{1} \to GF$ and one $FG \to \mathbf{1}$, and the composites $F \to FGF \to F$ and $G \to GFG \to G$ should be the identity. That's supposed to be represented in this picture.

So we want to be looking at $\infty - n$ categories with adjoints.

All $\infty - 0$ and $\infty - 1$ categories have adjoints. There's no condition. An $\infty - 2$ category with adjoints means that all 1-morphisms have a left and right adjoint. An $\infty - n$ category with adjoints means that all 1-morphisms have left and right adjoints and for all objects A and B, $C(a, b)$ is an $\infty - (n-1)$ category with adjoints. It takes a little bit to parse this, but think about the case of manifolds. It's telling you that the adjoint of a map is running the cobordism in the opposite direction. I said this in this way because I want to highlight some features. The maps should be an $\infty - (n-1)$-category, and eventually, the level n morphisms should form a topological space. So this is a possibly elaborate combinatorial thing and then eventually a topological space.

Right now I've expressed this notion of an adjoint as if it involves a lot of data. But what I'm heading toward is another point that a morphism having an adjoint is not adding a lot of data. It's not true, it either does or doesn't have it.

If I have an $\infty - 2$ category. That means to every object a and b, $C(a,b)$ is an $\infty - 1$ category. Every pair of maps, the transformations between them is a topological space. I can quotient down to π_0, and I can collapse this to what I call $\pi_{\leq 2}C$, which is an ordinary 2-category. The objects are the objects of C, the maps are $\pi_{\leq 1}C(a,b)$. I have to tell you what $\pi_{\leq 1}D$ is. The objects are the objects of D, and the maps are the set of path components of $D(a,b)$. Looking at a topological space, this would be the fundamental groupoid and $\pi_{\leq 2}$ is the fundamental 2-groupoid. So C has adjoints if and only if $\pi_{\leq 2}C$ has adjoints as an ordinary 2-category. Let me just say, that a morphism having an adjoint, it either does or it doesn't, it's not extra data.

Inside an $\infty - n$ category is an $\infty - 2$ category. A morphism either has an adjoint or it doesn't, it's not extra structure. You're supposed to believe, because I can define it in the other direction, so if I have a functor of $\infty - n$ categories, anything with an adjoint goes to something with an adjoint. Suppose C is an $\infty - n$ category. Then I can construct a largest $\infty - n$ category with adjoints $C^f \subset C$. You do this by starting at the highest level. You define, for each a and b in C, this is an $\infty - (n-1)$ category, so we have the largest one inside. That lets us define something I'll call C', where the objects are the objects of C but the maps are $C(a,b)^f$. So now we've taken care of everything except the 1-morphisms, so I take the subcategory where I take only the 1-morphisms with adjoints.

I have to make two other points and then I can do examples. This is very interesting. This is an elaborate condition. What the theory winds up telling you is that every functor from your cobordism category, because everything in Bord has adjoints, that factors through C^f. So if we're trying to classify these functors, we'll meet C^f naturally. That's one thing. There's another thing I don't like about the terminology here, this notion of having adjoints depends on n. This notion depends on n. So C^f depends on n. It should have been part of the notation. I could have considered this as an $\infty - (n+1)$-category and looked at the largest subcategory with adjoints, but that could be different.

Let's say I have an $\infty - 1$ category regarded as an $\infty - 2$-category. Then F having an adjoint is a natural transformation GF to 1, and this is a 2 morphism, so invertible, so G has to be the inverse to F. Crossing n, and ask that I have adjoints, this becomes a space, where everything is invertible. Even though that's not in the notation, I want to point out that this depends on n.

Let me repeat this warning as an example. If C is an $\infty - 1$ category regarded as an $\infty - 2$ category. Then all morphisms in C^f (as a 2-category) are invertible. I should have put an n in the notation somewhere.

Maybe it would be a good time to do an example. Let's go through one.

I want to start with C the 2-category of linear categories. Let's say they're of the form A-modules for some ring A. I have objects, categories, 1-morphisms (functors), and natural transformations (maps between the maps).

Back in the 60s there was this language of universes. All you had to do was make sure you couldn't state the Russell paradox. In the 60s it was easier to go from one universe to another. I don't know. These are, you can get around it that way, you just say a different level in the hierarchy.

I'm going to look at these categories. so objects are rings, maps are additive functors $A-mod$ to $B-mod$. I want to work out what C^f is. So suppose I have a functor $F : A - mod \to B - mod$ and I want it to have a right adjoint. It has to preserve surjective maps, sums, has an exactness property. If F is a left adjoint, it's determined by $F(A)$ because I can take a resolution. Call this M, so this is a B module. Then by functoriality M is an $A-B$-bimodule and $F(X) = X \otimes_A M$. To have a left adjoint means that my functor is given by tensoring with a bimodule. I might as well replace C by a smaller category where the objects are commutative algebras, a one-morphism is a bimodule, and the two-morphisms are bimodule maps, or intertwiners. If I look in this world, everything has an adjoint. I want everything to have a right adjoint too. So F is of the form $X \otimes_A M$ and now I've got a right adjoint G, if that's going to be a left adjoint too, then G must also be a left adjoint, G corresponds to another bimodule N. On the other hand, we already know what the right adjoint is. We know that $G(Y) = Hom_B(M, Y)$, so $Hom_B(M, Y)$ has to be of the form $N \otimes_B Y$, so taking $Y = B$ I find $N = Hom_B(M, B)$, and now look, this is a restrictive condition. If we take arbitrary Y then M must be finitely generated and projective as a B-module.

So that came from the condition that the right adjoint has a left adjoint. If I'd run it in the other way, I would find out that F has a right adjoint if and only if M is finitely generated projective as an A-module. So C^f in this case has objects commutative rings, 1-morphisms bimodules which are finitely generated projective as A modules and as B-modules. This is a very elaborate finiteness condition. The two-morphisms are bimodule maps. The adjoints are not necessarily the same.

So that's meant to tell you, it's meant to explain this F. It's an elaborate finiteness condition. Every map has some special structure. This sort of cuts things down into something very finite. We haven't used superposition and now there's something we'll get from that.

Inside manifolds we have the disjoint union, and inside any manifold, I can get a map, If this is M and I call this N, [picture], then I get a map from the cylinder on M, $M \otimes N$ to the empty manifold, to the identity. This goes into C. So this gives a map $V \otimes W \to 1$, the unit for the tensor structure. Then I get a similar thing, $1 \to V \otimes W$, and I can glue those together and if I go from $W \to W \otimes V \otimes W \to W$ or $V \to V \otimes W \otimes V \to V$ then these are isomorphic to the identity maps. This says that V is dualizable and W is the dual. In a symmetric monoidal $\infty - n$-category let me make a definition. An object V in a symmetric monoidal $\infty - n$ category is dualizable if it is dualizable in $\pi_{\leq 1} C$, which is an ordinary category with objects those in C and 1-morphisms equivalence classes of 1-morphisms in C. In $\infty - \infty$ categories it's not, it just keeps going on. In bordism what that will imply is that the equivalences are the trivial bordisms. In a space this is the fundamental groupoid. This is something, this notion of having a dual only depends on the 1-morphisms if there exists a W, maps like this, and equivalences. This is a condition on an object, not extra data. In manifolds, by crossing with an interval, everything has a dual. And the space of duals is contractible.

So now let's notice something. If W is the dual of V then maps from anything into W, $C(X, W) = C(V \otimes X, 1)$. Let's collect everything that we've said so far. Everything that we've said so far is, we have a bordism category, and I haven't been specific yet about this, and functors into a symmetric monoidal category, Z a symmetric monoidal functor. So it automatically has to factor through C^f. Now it also, every object has to have a dual. So it lands in the dualizable objects in C.

[This fails if you have geometry?]

There might be some evolution here, right. In a topological field theory it's the identity. This lands in the dualizable objects in C^f. That's different, these maps involved here have adjoints. So I'll call this C^{fd}, the fully dualizable objects, which is a full subcategory of C^f. Suppose that C is $Vect_k$. Then V is duualizable if and only if V is finite dimensional. That's another joke, it's supposed to be finite dimensional. Being fully dualizable is an extremely elaborate finiteness condition. There's one more twist. I want something that reflects structure in the geometry. What is a sensible map between two things here. This is a simple point but it's the last kind of collapse of structure. Every manifold, every zero manifold will go to a dualizable object, but I also have the dual, V^*. So if I have $V \to W$ and $V^* \to W^*$. I gave myself a map from $V \to W$ and $V^* \to W^*$, and that gives a map from $W \to V$. So these maps are inverses. So that means that every natural transformation is an equivalence. So I might as well, the objects are fully dualizable, but the morphisms are equivalences, although, well, I want to define this as a full subcategory and remember that the only maps are equivalences. The consequence is that the collection of TFTs with values in C is a space. I can take disjoint union, so it's a Γ-space. In the end we're trying to define a space.

So I want to state the first theorem here. Now let me talk about $Bord_n$, this is a refinement of the Baez, let me talk about $Bord_n^f$, framed, so the objects are 0-manifolds along with a trivialization of $TM \oplus \mathbb{R}^n$, and the k-morphisms will be M^k along with a trivialization of $TM \oplus \mathbb{R}^{n-k}$. There's a $GL_n(\mathbb{R})$ worth of points.

I was trying to avoid being precise, but a bordism should come with a map to the interval, a 2-morphism should be equipped with a map to an American football shaped region.

So a 0-manifold, how many objects have underlying space a single point. A point along with a trivialization, so $GL_n(\mathbb{R})$. I'v contradicted my notation. The first "theorem" is that $Bord_n^{fr} \to C$ is the space of fully dualizable objects in C (and equivalences between them). So let me say C_0^{fd}, meaning it's a space. I put this in quotes. This is a formulation of the Baez Dolan cobordism hypothesis.

It's proved for $n \leq 2$ and there's a detailed outline for all n. Jacob's talks are going to be about this. He's going to explain a detailed outline for how to do this for all n.

I want to talk about what it means and say what it gives for other structure. Their notion for dual, they didn't use this language. That was a lot of structure, not just a condition. It was seen as an object plus all kinds of data, but it's a property.

Now there's a very interesting corollary. Notice that $Bord_n^f$ has an action of $GL_n(\mathbb{R})$ since part of the data was the trivialization. Oops, I should have said that Z goes to $Z(pt)$. That gives me, given a TFT, it gives me a fully dualizable object. So GL_n acts on any one of these functors by precomposing. So the space C_0^{fd} has a natural $GL_n(\mathbb{R})$ action.

[This is a weird statement since it's a homotopy object. Is this an ∞-action?]

Yeah. I'm going to explain this in a minute but this is not obvious. We described this combinatorially, and out of this we have something with an action of a linear group. This is one of the mysteries of this theorem. I don't know how to explain this except for low values of n.

I'm going a little over time. Let's continue with the example. The GL_1 action sends the object to it's dual. I should have spelled out another example, where C is linear categories of the form A-modules. Then C^f was rings and bimodules where M was finitely dimensional projective over A and B. The objects that are fully dualizable, well, in C everything is dualizable. The dual of A is the opposite ring, in commutative it's just the ring. The map $A \otimes A^{op} \to k$ is A viewed as a bimodule, and the same thing in the opposite direction. So fully dualizable means that A is finitely generated projective over k and over $(A \otimes A^{op})$. If k is a field it's finite dimensional over k and I think the other thing says it's a finite product of simple things. In the dg version I would say it's perfect. So it would be smooth and proper.

So given, if you write down a product of simple algebras, there's a 2-dimensional TFT that sends a point to that algebra.

I don't want to study just framed manifolds. I wouldn't be able to get a value on most things, so we want to consider other structures. We have a manifold, and here's the thing classifying the tangent bundle plus $\mathbb{R}^{n-k}$, it's over $BGL_n(\mathbb{R})$. Now consider those maps factoring through an arbitrary space X, like BSO_n or $BSpin_n$ or $BGL_n(\mathbb{R}) \times S$ So I should have a map $\xi : X \to BGL_n(\mathbb{R})$ and call this $Bord_n^\xi$. So previously X was a point. Let me call it ξ. So

Theorem 1 *$Bord_n^\xi \to C$ is the space of lifts of ξ to $EGL_n\mathbb{R} \times_{GL_n(\mathbb{R})} C_0^{fd}$.*

That's the classification. If X is BG corresponding to a homomorphism G to $GL_n(\mathbb{R})$, then the space of sections are the homotopy fixed points for the G-action. I set this up to go through an example. I know from yesterday it's more tiring to be the guy sitting down.

Let's look at $n = 2$ and $G = SO(2)$. To give a two dimensional TFT for framed things I just had to give you an algebra, no more structure. I gave you no extra structure. Where I'm going to head is that the topological field theories are objects in a category satisfying a condition. Putting different structures, that will come from asking that this be a homotopy fixed point for the $SO(2)$ action. So we need to describe what this $SO(2)$ action is. How do I make, we're in this world of ∞ categories. What does it mean to have a map $SO(2) \times C \to C$? This is a category with a point for every point in the circle, a map for every path, et cetera.

That's equivalent to the category with one object, and the only data that's left is this path. So it's an object together with an automorphism. This structure $SO(2) \times C$ corresponds to a natural automorphism of the identity functor. So for every V in C I should have a map V to itself. So I need an $SO(2)$ action on the fully dualizable objcets. So that means there's a dual, so $\epsilon : V \otimes W \to 1$ has a left adjoint. By duality, that corresponds to a map $V \to V$. When the duality map has an adjoint, that's equivalent to having an automorphism. So in the case of dg-modules over a scheme, this becomes tensoring with the Serre object. This becomes tensoring with the Serre object. What does it mean to be a homotopy fixed point to the $SO(2)$ object? We have an isomorphism of T, a natural equivalence with the identity. In the case of complexes of coherent modules on a scheme, this is trivializing the Serre object, it's the Calabi Yau condition you run into in two dimensional field theories.

Let's go back. If I have a fully dualizable algebra, and I want to figure out, what does T correspond to? A map from A to A is a bimodule, and we know what it is. The duality may ϵ was given by A, and the adjoint is the linear dual of A. So the bimodule is A^*. So to make A into an $SO(2)$ fixed point (homotopy) is the same as finding a bimodule equivalence of $A^* \to A$. If A is a product of copies of k, a map like that is writing down where the idempotents go and this gives a Frobenius algebra. This was supposed to show that this gives all the general information that we're familiar with. This clarifies a little something. It tells you how this structure gets into place. To be for framed things, it's just an algebra, but to be oriented, this is a Frobenius algebra. You can get string topology by taking cochains on a manifold. You're going to hear a lot about this from Jacob.

2 Michael Douglas

I was asked to start not from the beginning but from undergraduate physics knowledge and to survey the long development that leads to topological and quantum field theory. I am supposed to talk about what questions that these ideas would work well with. I gave a long list of topics here more as a reminder to me. I think that the right starting point is with quantum mechanics. It makes most of the relevant points right from the start. I can go back further if you really want, but the usual starting point is Dirac's textbook, where he introduced the modern language for physicists. We can say that a quantum mechanical system has a Hilbert space $\mathscr{H}$ and operators $\mathscr{O}_i$, and then some $\psi \in \mathscr{H}$, which are rays, and then we have the expectation value $\langle\psi|\mathscr{O}_i|\psi\rangle$. So usually we have H a privileged operator. We can take $-i\frac{\partial}{\partial t}\psi = Ht$, that's H and then we can do $\psi \mapsto e^{-iHt}\psi$, and the eigenvalues of the Hamiltonian are energy levels. Time evolution would be multiplication by a phase.

This is getting more intuitive. In real life this might be an atom. We'd write down a wave function based on some number of electrons. We'd write a wave function $\psi(\vec{x}_1, \vec{x}_2)$. The eigenvalues would be the energy levels the electrons could be in. And there would be a minimal energy level. Then some set of eigenvalues, energies, what we would call excited states. These can wander off to infinity and then after they accumulate (at zero, or minus the energy of the atom with a factor of two put in), after which there is a continuous action.

This is an example that you can look up in a textbook.

Now what's a system which would have to do with TFTs? Instead of considering particles in space, you can consider particles moving in M. Then there is a simplest quantum mechanics associated with the manifold (M, g). From this there is a very standard construction, the Hilbert space is $L^2(M, \mathbb{C})$ and the Hamiltonian is Δ_g, the natural Laplacian from the metric. The inner product comes from the metric. I'm not going to motivate this.

The application I have in mind is a speculative one. Maybe the real theory of our universe. These make sense in 9+1 dimensions, or 10+1, so there's more dimensions, $\mathbb{R}^3$ cross other six or seven dimensional Ms. In practice there are other ways to get to this but they're low dimensional.

One way of motivating this from the baby example is to say that we have a limited amount of energy available. The first type of Hamiltonian is the Laplacian on $\mathbb{R}^{3n}$ plus some potential depending on the positions of the particles. Suppose that this potential has a minimum on some subvariety. Now a reasonable way to approximate at low energies is to say that you are constrained to the subvariety where the potential is minimized.

So anyway, let's take this as an interesting physical thing and start to play with it. Given the spectrum can you determine the M, and the basic answer is no, you really can't for anything above two dimensions. To a large extent you can play with the metric and adjust the lines to put them anywhere you want. If you look at it further, you notice other things, the simplest bit is Hodge theory. So look at $\Omega^*(M, \mathbb{C})$, and the natural Laplacians acting on forms. Then of course you have a quantum mechanics and the spectrum of zero value, the eigenvalues annihilated by Δ gives you the cohomology, so it's a huge amount of information. The nicest way to describe this information is to regard this quantum mechanics as a supersymmetric quantum mechanics.

Now as a physicist you wouldn't have thought of it very quickly. There is a class of physics problems that leads to this and that's with supersymmetry. I will simply observe that the space of differential forms, simplifying, is a fiber bundle over $L^2(M, C)$ which is generated by dx^i. So we can soup this up to a Clifford algebra so that $\{i_j, dx^i\} = \delta^i_j$. I could define this precisely but you can look it up.

Once you get this in this form, you can turn this into fermionic variables that square to zero. This is sort of the natural physics way into the forms.

[Yesterday we heard that you have a supermanifold and the functions on it are forms.]

That's right. So now I can produce natural operators whose anticommutators give me the Hamiltonian. I could write this whole thing out and I get $Q^2 = H$.

The other point to make about this is that, well, people don't start with Hodge theory, they start with functions and then introduce these things, or maybe not. There are many routes here. The natural thing underlying all of this is that we can not only act with the Laplacian, but act with other observable operators. These would be multiplying a wave function by a given function $\mathscr{O}_f \psi = f\psi$. Now once we have this in the game, we have an algebra of observables.

Now as opposed to the special problem, this gives a lot of information, it gives you all of the informations of the functions on the manifold. Ultimately this should be the primary thing. If you are given the algebra of functions and the Laplacian you can get the metric back, but more information is in the algebra of observables. Where I'm going is you can think, what is the space of these things. It's not true that all of the quantum field theories are associated to manifolds, but to some 0th order of approximation as some modified version of what are Riemannian manifolds.

Now what would have been the shortest, well, I wrote all of this and this is pretty much it for the foundations and now we can do the same sort of thing for a quantum field theory if we can find a commutative associative algebra. I'm going to show why this is not obvious in quantum field theory.

In quantum field theory there are natural sets of observables. What is true is that it doesn't require very many wave functions. So you have the first k eigenfunctions. If you can add, multiply, and take limits, those give the whole algebra.

That was the reminder of quantum mechanics. So now quantum field theory. A good definition is this functorial definition which I will basically repeat, first for the setting I described. A QFT is a functor $\Sigma_d \to$ some linear category. For $d = 1$ this is quantum mechanics, points go to a Hilbert space, and then t goes to the operator e^{-itH}. With graphs you get correlation functions for ψs. (You think that your Hilbert space is an algebra). So L^2 is not an algebra. I don't know whether anyone has said this carefully. Kontsevich has described this to me. I think it's [unintelligible]easy to answer this question. If you put in finite lengths coming into a vertex, if in the previous notation you took, well, we change our time so as to get rid of the i and so we can take $e^{-t_3H}[(e^{-t_1H}\psi)(e^{-t_2H}\psi)]$ and define a multiplication.

[Enlarging the functor to act on graphs and now you get rings]

One goes quickly from a Lorentzian manifold to a Euclidean manifold. I'm not going to draw the next step. Until you start asking about gravity, physicists do [unintelligible]

This is the kind of setup we might try to realize in greater than one dimension. It's complained that there is no very satisfactory condition for $d > 1$. I think what's best is for me to wave my hands. So the, uh, you could try to construct this correspondence. Then you would be, you can be quite explicit about the time evolution and even the product (not worth the trouble). What Im leading up to, loosely speaking there's this path integral, we have some sort of paths from $\Sigma_d \to M$. We weigh these by some sort of action. I'll write the dreaded path integral

$$\int [dX] e^{-S(X)}$$

Here in $d = 1$, let's talk about S^1 and the interval. In both cases, they want to be maps into M. So let's take $M = \mathbb{R}^n$ with the Euclidean metric. Then these maps, the meaning of this integral.

The meaning of the integral is an expectation under a probability. There should be a preferred state in H called the ground state ψ_0. If I have an observable, then $(\psi_0, \mathscr{O}_f \psi_0)$, then $\int [dX e^{-S(X)} f$ is $E[\mathscr{O}_f]$. The goal is to find the expectation value is going to be the expectation of that function under some nice probability measure.

I will consider an f that takes one value at one point. In the quantum mechanics I want to literally define $(\psi_0, e^{-tH} \mathscr{O}_f \psi_0)$. Now we're taking the function f as a function of the value of this map at some preferred point in our discussion. The X is a loop or a map from the interval, when I write this out explicitly, say, $S = \int dt (\frac{\partial}{\partial t} X(t))^2$. The ψ_0 has to do with the boundary conditions. Under the integral I choose maps weighed by appropriate wave functions.

I'm afraid of this getting a little intricate. The standard choice for ψ_0 is to take the limit of extending the time evolution for infinite negative time and concentrating on the minimal eigenvalue. It would be a constant function in this case. You can write all this out, and then this measure you can make rigorous. This is a variation on Weiner measure.

So that's the case where the problem is well understood mathematically. The majority in defining $d > 1$ says you should follow the same approach with higher Σ. Let's try to do a similar thing for the cylinder, and take $d = 2$. Now we have X, and we can take this to be the Euclidean cylinder. Then $X : \Sigma_2^t \to \mathbb{R}^n$. You can develop this as rigorously as you like, so you take the case where the target space is Euclidean and you're working with the norm squared of the gradient.

$$\int [DX] e^{-\int_{\Sigma_2^t} ||\partial X||^2}$$

The standard treatment notes that the exponent is quadratic and that it's the exponentiation of the Laplacian, which can be diagonalized. Formally we have an infinite dimensional Gaussian integral, and you can do it. If you want to learn about how the physicists do things, read about this. So now you can say very precise things about the corresponding measure, but the difference from $d = 1$ to higher d is that the measure is not supported on continuous paths above $d = 1$. Integrate it once and the time evolution of that integral will be continuous. The standard physics way of talking about this thing is to write a two point expectation value. At a point $(0, z)$, you can integrate this with $X(z)X(0)$ and you get something that is approximately $-\log |z|$. The general formula is $|z|^{2-d}$ for $d > 2$.

What you will now will be led into is that you want to start to smooth these functions. This is now the story for Euclidean M and these are well-defined results. There is a well-defined field theory of this type. If you have a non-flat metric on M, as I mentioned earlier, we could in principle weigh the integral by another function V, there's a variety of interacting theories, anything where it's not a Gaussian integral. It's difficult to define these theories but in the simplest case, called perturbation theory, you can trace it back.

The key insight, from Wilson around 73, is the renormalization group. Let's make a simple-minded approximation that avoids the problems with multiplying. Now instead of taking functions on a manifold, take a finite approximation, where we have finite differences and a lattice. You should think about this from the point of view of a large scale observer, the real system could look like this, but at a large scale it looks fuzzy. What you're interested in is the limit where this looks like the field you've postulated. You can define the value of X at one point as the average near the point. Then you measure your averages. Take a two by two and average those. Try to average out the three differences. Then we try to estimate what the measure looks like as a function of the average variables. The really key point, which Wilson argued, is that in many circumstances, in particular the physically realizable ones, the actions you get out look just like the ones you get from the larger scale. There's some sort of process that moves from L^1 to L^2. You will get something that looks so much like the old one that you have fixed points of the original one.

Here's a space of actions. What defines a point here. You presuppose a mainfold and a coordinate system. Here I've given a function and I can specify more things, higher powers of the gradients, but once you start making this transformation, everything comes down to a finite dimensional space of actions.

So a good deal of this problem, understanding the space of these things is easier than understanding the flow, that integrates up to this map, and in particular to look at fixed points of this map.

[Too much physics. I stop.]