Focused Research Group Workshop

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September 11, 2008

1 Teichner

[We have two and a half hours until tea. If Peter wants a break, he can, or we can just go through.]

Beyond Segal's axioms. Before this talk, I wanted to remind you that we meet again January 6-10, 2009, in Berkeley. This should be a nice time in Berkeley, January is a good month. This is your last chance to object to this schedule, or I will begin to talk with people about who we will schedule.

I have four points I want to get to. You're already very good at interrupting so please do so.

Let me remind you of Segal's notion of a CFT, and then I want to add, well, I guess I'll start by giving our precise version. All I'm saying is joint with Stefan, and some of it later will be joint with Chris. We used the notion of internal categories. And this conference or workshop is a lot about language. We wanted to make a space of two dimensional field theories which would classify TMF. It took us many years to get closer, and with the language of internal categories we think we have something that looks really good. We can compute in dimension zero and one. If you can complete this to dimension two, well. This internal category, it is very interesting, you get cohomology theories like we expected, but we wanted to present it for topological field theories. Even though we started out trying to understand quantum field theories. Since most of you are interested in topological field theories, we can compare notes. We'll be concentrating on the TFT part. I'm just going to be topological.

The first landmark, I'll put it over here, are these internal categories as a language. This will include the notion of what a smooth functor is. Segal's field theories are certain functors, and here it helps to think that they are smooth.

Then we will add bells and whistles. For example we can pass to supersymmetry. It will be very easy once you have the language. We did not like this, but we were forced to it because we only had the cohomological properties we desired in the supersymmetric case. We want to talk about TFTs over a manifold X. This is important if you want to think about spaces or Abelian groups. That's like a map from X into the space of field theories.

[On Thursday we have a physicist coming whose problems involve needing a space of all theories.]

Then also twisted TFTs. When I do the Segal thing I will talk about central charge zero, for simplicity. Without twisting, you only get 0 cohomology. We use this because of the connection to twisted cohomology. It comes up in physics because the natural examples are twisted, not field theories in the naive sense.

This is my rough outline. I don't know how much I should say about this. A Segal CFT (that's what Segal wrote about, but we'll erase the C and make it a T in a second), he's going to study symmetric monoidal functors. There is this great fact, that two conformal things, you can glue canonically along any diffeomorphism. It turns out it's not a problem to have things overlap a little bit. I'll just say QFTs. It's a symmetric monoidal functor from $2-RB$ to TV. The first is the Riemannian bordism category of 2 and 1 dimensional manifolds. So surfaces and circles. The next one is the category of topological vector spaces, which can be clarified to topological vector spaces (Frechet or Hilbert are possible specializations). The monoidal products are disjoint union and tensor product. The second landmark is to make this precise using internal categories. The R stands for Riemannian, and the B stands for bordism category. Really, maybe it should be Lorentzian. In math we always do the weak rotation, we make the time imaginary and then avoid a lot of problems. To make CFT you replace Riemannian with conformal, and for TFT with smooth. The tensor product is the projective tensor product. In algebra, any bilinear map $V \times W \to Z$ factors uniquely through $V \times W$ as a linear map. This should be true for continuous bilinear maps in this setting, and sometimes you need to complete this. So $V^* \otimes W$ are the trace class operators (V, W). For the Hilbert space tensor product you get Hilbert Schmidt operator.

If I take $C^\infty(X \times Y)$, that's the projective tensor product $C^\infty(X) \otimes C^\infty(Y)$. Maybe I should say complete here. So I want to talk about complete topological vector spaces, and then the tensor product, I want to complete it.

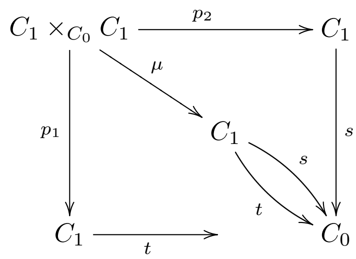
The third landmark is, I wanted to explain everything for dimension zero. Stephan in his talk will do everything in dimension one. Mike and Jacob will explain locality. This is roughly the idea that this should be a two-functor. I'll skip the higher category theory.

Both of the categories involved in my functors are internal categories, so this will be, maybe I should remind you of internal categories. If I have, I am starting with an easy thing, an ambient category $\mathscr{A}$. I want to define a category in $\mathscr{A}$. (for $\mathscr{A}$ Sets, this is the notion of small categories).

Definition 1 *A category in $\mathscr{A}$ consists of $C_0, C_1 \in \mathscr{A}$. Then I have two morphisms, source and target, $C_1 \to C_0$. I want a composition $C_1 \times_{C_0} C_1 \xrightarrow{\mu} C_1$, so I need this pullback to exist:*

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{p_2} & C_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array}$$

This pullback is the universal thing so that if you have, well, it satisfies the pullback thing. The two properties af a category. What the property is, well, after the composition you know the target.



Then there is also an associativity relation for $C_1 \times_{C_0} C_1 \times_{C_0} C_1$. We don't have a unit yet, you can put that in with a morphism C_0 to C_1. Some things are easier.

Let's look at some examples to motivate. I wanted to give the main example for a d dimensional TFT. So you talk about the d dimensional bordism category. This is a category in A. So I need $(d-B)_0$ and $(d-B)_1$, and then these structure maps. For $(d-B)_0$ take the category of smooth closed $d-1$-manifolds. It's not a small category, I want this to be a groupoid. The morphisms are diffeomorphisms. For QFTs, (this is the first run at the definition), I should have said closed Riemannian manifolds with isometries. The objects are bordisms, compact, d-dimensional bordisms and their diffeomorphisms. I'm trying to give you a rough picture. What is this, really. I need a d dimensional compact manifold, with an incoming and outgoing boundary [no orientations], $Y_t^{d-1} \hookrightarrow \Sigma^d \hookleftarrow Y_s^{d-1}$. with diffeomorphisms.

This isn't quite correct. The source and target morphisms (functors) are restriction to s and t. We will need a collar in a second and the boundary of Σ should be the union of these two pieces.

Now we need a composition functor $(d-B)_1 \times_{(d-B)_0} (d-B)_1 \to (d-B)_1$. This doesn't work. I don't know what the smooth structure will be. I have a topological manifold but no unique smooth structure. This is the point where Segal didn't want to throw in collars, he chose conformal things. Even with TFTs, the correct thing is to throw on collars. The topological and quantum are more difficult than conformal theories.

I have to be a little more precise. That's my second landmark, let me defer it. Ignore the gluing problem for a second. There is another problem. [Try gluing two things along a point, you need one parameter for the C^1 structure, two for the C^2 structure, and so on.]

So this is not associative, it's associative up to homotopy. So we want to say there is a canonical associator satisfying the pentagon. This should not be a category, but a two category, it should be strict. There is associativity to whatever you want. So small categories is a two-category with natural transformations. I can define over what a category in $\mathscr{A}$ is.

Now I have to make some choices. A diagram that ends in C_0 commutes strictly. The associativity is not strict because it ends in C_1. There's an associator which satisfies the pentagon, a natural transformation. The ambient category actually was a two-category, and now this is a beautiful example except for the gluing. Now let's put in, this is one example. The other example I should give is vector spaces. So there's a second example. I'm going to take a break, don't worry, I can't stand here for two hours.

I want to turn TV into an internal category. What I'm doing here you can probably do with any category that you like. So let TV_0 be the category of complete topological vector spaces with isometries, and TV_1 is the category of continuous linear maps $(V \to W)$ and commutative diagrams of these $TV_1(V \xrightarrow{f} W, V' \xrightarrow{f'} W')$ is the set of diagrams

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \alpha\cong}\downarrow & & \downarrow{\scriptstyle \beta\cong} \\ V & \xrightarrow[f']{} & W' \end{array}$$

If I give you f and f', that gives two objects. If I give you two isomorphisms, if this commutes it's a morphism. This is a formal trick, you take a category and make it an internal category. That implies that in the naive example we'll get diffeomorphism invariants.

A d-dimensional TFT, this is still the first run, is a functor between internal categories $(d - B) \to TV$. These are now internal categories, maybe I should add a star, and these should be internal functors. So having two of these, say C and D. I want to explain what an internal functor from C to D is. Most of it you can imagine. I'm going to have an $f_0 : C_0 \to D_0$ and an $f_1 : C_1 \to D_1$. There are two diagrams ending in D_0 that commute having to do with source and target

$$\begin{array}{ccc} C_1 & \longrightarrow & D_1 \\ \downarrow & & \\ C_0 & \longrightarrow & D_0 \end{array}$$

and another for composition:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{F_1 \times F_1} & D_1 \times_{D_0} D_1 \\ {\scriptstyle \mu_C}\downarrow & & \downarrow{\scriptstyle \mu_D} \\ C_1 & \xrightarrow[F_1]{} & D_1 \end{array}$$

which commuties up to F_2. This will satisfy a hexagon. The associators in C, those in D and the Fs should fit together.

Let's go back to Dennis' question. A field theory is a functor from this to that internal category. For every $(d-1)$ manifold I have a complete vector space, for every diffeomorphism an iso, for every bordism I have a continuous linear map, and for every diffeomorphism I get a commutative diagram. This implies that if I have a diffeomorphism that is the identity on the boundary, a diffeomorphism rel boundary tells me that f and f' are the same, if you have a diffeomorphism, some diagram commutes. Diffeomorphic d-manifolds give the same linear map. Once we put the bells and whistles you will see this is the only way to do it.

I have now a first run on a d-dimensional TFT. I'll unravel it in dimension zero and one, we could do this now, it's boring. It will be interesting over X and with supersymmetry and so on.

There's a couple of thing's I'd like to get straight. For our ambient category we used the category of categories. We have two functors and what I was saying is that the functor on the morphism level says that the diffeomorphisms go to certain identities. It puts in the condition that the associated linear maps are equal. This will give an Atiyah Segal TFT. This is the same except that Atiyah Segal is monoidal. You have to change the ambient category from Cat to symmetric monoidal categories. You just use disjoint union and the (completed) projective tensor product, just realizing what you wrote down has more structure. Then you can require the functor between the two, well, the functors are functors of internal categories, so functors of symmetric monoidal categories. Okay, so now the next thing I want to do is, I want to go to $smCat/Man$. What I want to do is, this is now the version I want to think of, the usual notion modulo the gluing of bordisms. Now we want to say what a smooth functor is. A smooth d-dimensional TFT is in an internal functor in $smCat/Man$ from $(d-B)^{fam}$ to TV^{fam}. So now I should explain to you a certain ambient category. Man is the category of manifolds (smooth) with all smooth maps. Then Cat/Man is a new two-category, but basically the objects are $C \to Man$, a category with a functor to Man which is a fibration. This is not a big issue, the examples are Grothiendieck fibrations.

I have to give you two examples $(d-B)_0^{fam}$ and $(d-B)_1^{fam}$. This category $(d-B)_0$ is a manifold S and then a family of $d-1$ manifolds over S, so a smooth fiber bundle of closed $d-1$ manifolds. What is a family, it's a fiber bundle where the fiber's any manifold. A morphism for pairs would be a map which fiberwise is a diffeomorphism. I hope I don't have to explain $(d-B)_1$, it is the same thing but over S. That defines these two categories, and then I claim that what we had before, these functors s and t, and the diagrams make sense over this larger ambient category. So TV_0^{fam} is the category of vector bundles over manifolds (or sheaves or some type of sheaves, we're not sure). In the cases we'll discuss today we'll use vector bundles. Because these are vector bundles over manifolds, we can forget to the manifold, so this is in Cat/Man, and then we need a symmetric monoidal structure. If you have two bundles, this is the external tensor product, where you take the cartesian product of the base and the tensor product of the fibers. In the domain I take the cartesian product in the base and the disjoint union in the fiber. A smooth functor associates a smooth family to a smooth family. One way to characterize a smooth function is something that sends smooth functions to smooth functions. This is the most naive version, the Yoneda lemma version. A manifold gives you a functor from manifolds to sets, so a generalized manifold is a functor from manifolds to sets.

A smooth functor is a natural transformation of these functors. I can take $Man \to Fun(Man^{op}, Set)$, and if I have an infinite dimensional manifold in M^{op}, I can still say what a smooth map from a finite dimensional manifold into an infinite one is. If I started between two manifolds, on the other side it's a natural transformation of functors. I hope you have a rough feeling. A smooth TFT is a functor from the family version of bordisms to a family version of TV

At a point you get the previous discussion. A smooth functor is a whole different beast.

A smooth TFT over X has as its bordism category an additional map from the total space of the bundle, smooth, $\varphi : Y \to X$. The maps agree on the boundaries.

The last thing I want to replace smooth with supersymmetric. So now you change manifolds to supermanifolds. Oleg will ask what that means. It's an ordinary manifold $(M^p, \mathscr{O}_M)$, with a sheaf of commutative superalgebras over $\mathbb{R}$, assuming, well, $\mathbb{Z}_2$ algebras with a sign thrown in, and these are locally isomorphic to $C^\infty(\mathbb{R}^p) \otimes \wedge^*(\mathbb{R}^q)$. The dimension is $p|q$. I'm almost done with my final definition. In a supersymmetric one, the base is a supermanifold, the fiber should be as well. Now it's a $(d-1|\delta)$ dimensional fiber. In our world we will say $\delta = 1$. You need a superfamily version for vector spaces.

I think I am finished with number one, let me skip number two. To make things precise as far as gluing is concerned, you need to expand the $d-1$ manifold into the germ of a d-manifold. The geometry only happens in dimension d. The definition should be that [unintelligible]and the geometry should live in dimension d. As long as your geometry glues upon open sets, the definition should work.

We orient just the normal bundle, coorientation. These guys could be nonoriented and the ambient guys are nonoriented, but you have a coorientation.

Ok, I'm sorry that this is a lot of definitions. Let me do this in dimension zero, I promise that it's fun.

Let's start with the very first observation. There is a unique -1-manifold. A TFT associated something to $d-1$ manifolds. The empty set is the monoidal unit. Any functor, it should be the monoidal unit up to an iso, let's ignore that. So this should be the ground field, I will take $\mathbb{R}$. So F_0 has no information. There are no diffeomorphisms of the empty set that we have to discuss. If I add families of things with fiber the empty set, it's still the empty set. All we're down to is the $F_1 : (0-B)_1 \to TV_1$. When I go down my list, I didn't have the symmetric monoidal structure. Now I'm talking about the category of closed 0-manifolds, because it's a bordism between the empty set and the empty set, along with diffeomorphisms. But it's symmetric monoidal. It's a bunch of points, because this is symmetric monoidal it's determined by its image on one point. So now I can instead study functors on the category of connected zero-manifolds. So this consists of the point and the identity morphism to TV_1. This is the business that the functors commute. The source and target should commute. This is a map from $\mathbb{R}$ to $\mathbb{R}$. The morphisms were commutative diagrams. There's a morphism if and only if the two numbers are the same so we take a point to a real number. This is pretty boring.

Let me put X in there to make it a little interesting. Closed 0 manifolds with a map to X, and when I specify the connected guys, the objects are points in X with the identity. Now I have a functor which to each point gives a functor. So for $smCat$ over X I get $Maps(X, \mathbb{R})$. When you changed the ambient category to $smCat/Man$ I get $C^\infty(X, \mathbb{R})$. The objects are families of points. So what do I associate? To it I associate not just one real number, but a function over S. This was pairs $(S, f : S \to \mathbb{R})$. In the family version this function is smooth.

It's the representable category represented by $\mathbb{R}$. Similarly here, this category is represented by X, it's an object S and a smooth map to X. So this turns out to be a smooth map from S to $\mathbb{R}$. In the supersymmetric case we get $\Omega^0_{closed}(X)$. With twisting you get higher forms. This is form $d|\delta = 0|1$.

Let me restrict to the connected case. I have a stack represented by X and one by $\mathbb{R}$. Now I take a superpoint. Now I have $S \times \mathbb{R}^{0|1}$ over S. The target category doesn't change. Again it's the stack represented by $\mathbb{R}$ on the site of supermanifolds. So the domain is $S \times \mathbb{R}^{0|1} \to X$ over S. There's a beautiful adjunction formalism that $SMAN(S \times \mathbb{R}^{0|1}, X) \cong SMAN(S, \pi TX)$ where this is the supermanifold associated to TX. So this is associated with the tangent bundle. So that means that a functor would take, well, then I have to have Aut or Diff of $\mathbb{R}^{0|1}$. So what my functor does, F_1 is a function on πTX. It associates to each map on πTX a map to $\mathbb{R}$. These are the sections of $\wedge^*(T^*X)$ so it's $\Omega^*(X)$. A super point has diffeomorphisms. It has an odd direction, the translations, and an even direction, the dilations. On the right you have only identities, so you have to be invariant under the diffeomorphism group to have a TFT. More geometry reduces the symmetry group. Put that subgroup there instead. The fact that this functor sends these symmetries to identities tells me I should take forms invariant under these. If I had a structure with no symmetries I'd get all differential forms. The $\mathbb{R}^{0|1}$ action is generated by d and the $\mathbb{R}^\times$ action gives the grading, so then if I want to divide both out I get closed zero forms. We like the Euclidean ones where you get dilations but no translations and you get all closed forms of all degrees.

This is the final punchline, with TFTs you have a lot of symmetry and you're left with a tiny space. The de Rham algebra is functions on the supermanifold of superpoints in X.

In dimension zero, if you want more symmetries, put $\mathbb{R}^{0|k}$ you can use whatever subgroup you want. You can work out what things are. I should quit, I talked way too long.

2 Stephan

I'm charged with showing you the one dimensional case, distinguishing between closed things and things with collars. The other things that I will use, well, let me repeat some things. If you have a vector bundle E^q over N^p, then he constructed π of this, which is N along with $C^\infty(N) \otimes \wedge^*(E^*)$, which is a supermanifold of dimension $p|q$. In particular we have the smooth functions of πE are just the sections of the exterior algebra bundle $\Gamma(\wedge^* E^*)$. If you look at πTN you see $\Omega^*(N)$. The other important thing, let me write πTX. For any supermanifold, morphisms $SMAN(S, \pi TX)$ can be identified with $(S \times \mathbb{R}^{0|1}, X)$. Morphisms are in terms of sheafs. Here you can say they are maps of algebras compatible with the grading. If you have a map from S to T it's the same as a continuous map from the functions of the target into the domain.

Let me recall the definition of a $d|\delta$ TFT over X. This is an internal functor $E : (d|\delta - B)^{sfam}(X) \to TV^{sfam}$ with the ambient category $\mathscr{A}$ being the category of symmetric monoidal categories over supermanifolds.

I want to spell that out. It's like a functor on the object category and on the bordism category. It's $E_0 : d|\delta - B_0^{sfam}(X) \to TV_0^{sfam}$. In the domain, the objects are a bicollar, a $d|\delta$-manifold and in there a $d-1|\delta$ manifold Y^c inside Y and then a map $Y \to X$. That's without being in a family. That just meant that the Y is now a fiber bundle so that the restriction to Y^c is also a fiber bundle, with typical fiber looking like Y. Applying this functor, then, keeping the Y and the f in the notation, will give us a vector bundle $E_0(Y, f)$ over this supermanifold S. Let's see, let me try to put this in. You want to say that the vector space for (Y, Y^c), that vector space varies smoothly as you deform this, and here the best way to express this is to say that you have a functor that does that for families. We have a similar thing for d dimensional bordism. To these we want to associate maps (in families). For me I need to go back to the non-family case to ground myself.

So E_1 goes from the $d|\delta - B_1^{sfam}(X) \to TV_1^{sfam}$ and the picture for a typical fiber looked like that, what we see is a bordism which is a little bigger on the ends since it includes the bicollar around the objects, both of them, Y_2 with Y_2^c in it. You get an ordinary bordism by cutting away the things that hang over the edges on both sides. What does that mean? We have not just the inclusions of Y_0 and Y_1 into Σ, these are fibered over S, they're all fiber bundles, and we have a map to X from the total space extending the maps from the Ys. What do we associate? A linear map between vector spaces (in families) so a vector bundle map. So we get $E_1(\Sigma, p) : E_0(Y_0, p_0) \to E_0(Y_1, p_1)$

Let me sort of extract from here how I'll get a differential form. Again, let's look at the case $d|\delta = 0|1$ with E a $0|1$-dimensional TFT over X. There is no information in E_0 All the beef is in the second piece. The only interesting case is $S \times \mathbb{R}^{0,1}$ Then I need a map $f \to X$ and I can apply E_1 to it. So this is a map of trivial vector bundles, so just smooth function on my parameter space.

Now I want to specify S to get a differential form. So take πTX and so we have $(S \times \mathbb{R}^{0|1|} \xrightarrow{f} X) \leftrightarrow S \to \pi TX$.

Then you can show eventually that you live in $\Omega^* X$ and you have a closed form and then it is degree zero.

I don't want to go through the preliminary steps. I want to discuss $d|0$ and $d|1$.

What kind of objects do you have here? What kind of thing is it? These are defined in terms of functors. This is really a category. What is a natural transformation of these? I don't want to go through that. This will eventually be equivalent to the category of vector bundles with connections over X. Morphisms are isomorphism that preserve the connection.

Okay, so let me show you how to produce this one here. Start with a vector bundle with connection, W over X and connection ∇. You want to construct a one dimensional TFT E_0, E_1 over X. Here I mean finite dimensional bundles. These bundles should be finite dimensional. At a point you should see an ordinary TFT and the states will be finite dimensional. Find a dual vector space, and this even works in dimension one.

So $Y^c = S \times \{0\} \hookrightarrow S_x \times (-\epsilon, \epsilon) = Y \mapsto f^*W|_{S \times \{0\}}$. So now for E_1 I have $S \times (-\epsilon, \epsilon) \hookrightarrow S \times (-\epsilon, 1+\epsilon) \hookleftarrow S \times (1-\epsilon, 1+\epsilon)$ over S. So I want a map $f^*W|_{S \times 0} \to f^*V|_{S \times 1}$. Well, I have a connection which I can push. I can extend a section, move it to be parallel in the $\mathbb{R}$ direction, and then restrict it on the other side.

So these give you a parallel translation. If you have a parallel translation, you differentiate to get the covariant derivative. To show the map in the other direction was much harder. This was supposed to be the easy direction. Take the bundle, pull back, and use parallel translation.

Now we want to be given a one-dimensional TFT (E_0, E_1), we want to construct a vector bundle W over X and a connection on it. It's easy to extract the bundle by applying the functor E_0 to the universal case $X \times (-\epsilon, \epsilon) \to X$ viewed as a bundle over $X = S$, with $Y^c = X \times 0$. So you apply E_0 to this object and it spits out a vector bundle over the parameter space X. The first thing that you should notice is that the field theory gives you a bundle over a slightly bigger space.

This will be $\mathscr{G}$, the smooth mapping space from $(-\epsilon, \epsilon) \to X)$, so X sits inside this as constant maps. This is infinite dimensional but I can view it by the Yoneda embedding as a generalized thing. So I can take $\mathscr{G} \times (-\epsilon, \epsilon)$ over $\mathscr{G}$ with $\mathscr{G} \times \{0\}$ as the core with a map to X. The restriction to X, the constant maps, is consistent with the previous definition.

The claim is, well, let me do a little bit of notation. Suppose you have a map $f : (-\epsilon, \epsilon) \to X$. Then (I should have made $\mathscr{G}$ the space of germs of such) $G(f, s) \in \mathscr{G}$ for $s \in (-\epsilon, \epsilon)$. The point s corresponds to 0, this is $G(f, s)(t) = f(t - s)$. I can wonder, what is the relationship between the fiber $W_{G(f,0)}$ versus $W_{f(0)}$ where this is viewed as the constant path. One is constant, the other is not constant. The claim is that there is a canonical isomorphism between them. The construction is to pick a point s which is between 0 and ϵ and then you compare the fiber of the germ at $f(0)$ with the fiber of the germ at the point s. Then you compare that with a different path $W_G(f', 0)$ and $W_G(f', s)$. So to find f', it's $f \circ s$. I want it to be on the diagonal and then from some point on it should be zero. You see that the germ of this at zero is just the constant path, whereas the germ of f' has the same germ as f. Now I want to use the functor E_1, so I can use the path 0 to 1 as a bordism and evaluate my functor E_1 on f restricted to the interval $[0, s]$ and that gives an isomorphism. Of course with some choices, but if I make a different choice, you can show it does not change the bordism. It's independent of the choice of S and the shape of this graph. You can reconstruct over the larger space of germs. Now I'm in the position to say that what I'm gotting is like an ordinary parallel translation. So far you couldn't do that because you caould have different fibers.

Now I want to switch gears and talk about the supersymmetric version of that. This was $1|0$ and now I will do $1|1$. We want to say that such TFTs over X is equivalent to the category of superbundles over X equipped with a flat Quillen superconnection. First I should explain what a Quillen superconnection is. Suppose you have a $\mathbb{Z}_2$-graded vector bundle over X. Then you can look at differential forms with values in W, so $\Omega^*(X, W)$ which is a module over differential forms $\Omega^*(X)$. Then a Quillen superconnection on W is a linear map from this space into itself $A : \Omega^*(X, W) \to \Omega^*(X, W)$, linear, odd (total grading of W and Ω) such that $A(\omega s) = d\omega \wedge s + (-1)^{|\omega|} \omega A(s)$

Examples: Suppose that ∇ is an ordinary connection on W. Then you go $\nabla : C^\infty(W) \to \Omega^1(X, W)$, and then you can extend it to all forms, using the Liebnitz rule. Every element can be used starting with a section of W and a differential form. So you can extend and it satisfies this property, and an ordinary connection gives you a superconnection, canonically and uniquely. Here's a more interesting example. Suppose that A_0 is an odd endomorphism of W. Then you can extend it by the identity on the forms and take as A your $A_0 + \nabla : \Omega^*(X, W) \to \Omega^*(X, W)$. This was the kind Quillen was most interested in. He wanted interesting differential forms on the base, if you have a category of spin manifolds, you can look at the infinite dimensional vector bundle whose fiber is the spinors. There's a natural connection from the Riemannian structure on the total space. This is not good for a K theory element because it's infinite. You have a Fredholm operator on each fiber so this gives the index bundle and you get a K theory element. So Quillen took something like that where now A_0 is the Dirac operator.

Now I want to explain what flat is. You can define what curvature is, and you basically define it by taking the square $A^2 = 0$.

So given a vector bundle W with flat super connection, you want to construct a $1|1$-dimensional TFT (E_0, E_1). So this will be $E_0 : 1|1{-}B_0^{sfam}(X) \to TV_0^{sfam}$. So you have $\mathbb{R}^{0|1} \times (-\epsilon, \epsilon) \to S$ and out of this data you want to construct a vector bundle over S. A map from $S \times \mathbb{R}^{0|1}$ to X you can get $\hat{f} : S \times (-\epsilon, \epsilon) \to \pi TX$, over which we have a vector bundle, a projection p to X. To describe a map between supermanifolds, it's enough to say what p^* is from $C^\infty(X) \to \Omega^*(X)$. So pull this bundle back by the projection map and $\hat{f}$ and then restrict that to $S \times \{0\}$. It's the same technique I used in the previous situation, but πTX plays the role from the last time. An object here is the germ of a path in πTX.

Now how do I construct E_1? This should go from $1|1 - B_1^{sfam} \to TV_1^{sfam}$. Instead of taking the germ of a map like that, take a map $S \times \mathbb{R}^{0|1} \times (-\epsilon, 1 + \epsilon)$ mapped into X over S. You know you restrict E_0 to the ends, and you get $\hat{f}^* p^* W|_{S\times\{0\}} \to \hat{f}^* p^* W|_{S \times \{1\}}$. So now we need a connection ∇^A, which takes place on p^*W. This will be made out of the superconnection.

I need to show you how you can use a flat superconnection to make a connection on this bundle. It will be the flat connection.

The first thing I want to point out is that it's absolutely important that this be flat. There are plenty of isomorphisms which are the identity near zero and one but that give you very different things elsewhere. In terms of paths in πTX. I am doing translations along different paths which are homotopic and agree on the ending collars.

So let's construct ∇^A. Geometrically we have $C^\infty(\pi TX, p^*W)$ which correspond to $\Omega^*(X, W)$ and $C^\infty(\pi TX)$ to $\Omega^*(X)$. What is a connection on p^*W formally? It's a covariant derivative everywhere. It's a map $\nabla_U : C^\infty(p^*W) \to C^\infty(p^*W)$ for any vector field U on πTX so that Leibnitz holds. What is a vector field? you have a lot of interesting things, like d and ι_V and their bracket is the Lie derivative $\mathscr{L}_V$. So $\nabla^A_d s = A(s)$. In other words, the superconnection defines the covariant derivative on πTx but only in the direction of this very particular vector field. How about $\nabla^A_{\iota_V} s$? What is s? It's a differential form with values in W. For that thing it's easy enough to contract even with values in W and so we can contract $\iota_V s$. We want a flat connection. How about the curvature in the direction $R(\iota_V, \iota_V) := [\nabla_{\iota_V}, \nabla_{\iota_V}] - \nabla_{[\iota_V, \iota_V]}$, and it turns out that both of these pieces are zero. Since these things depend quadratically on V, the curvature in the direction ι_V, ι_W is zero.

How about the curvature $R(d,d)$? that's the graded commutator $[\nabla_d^A, \nabla^A, d] - \nabla_{[}^A d, d]$. So the second piece is zero. and this is $\frac{1}{2}A^2$ and that's zero since the connection is flat.

That doesn't define the covariant derivative in all directions. What about in the direction $\mathscr{L}_V$? What about $[\nabla_d^A, \nabla_{\iota_V}^A]$. That makes it automatic that the curvature should vanish in these directions. These span locally and we've defined the derivative in all directions. Now I can use it on a field theory. Okay, so that's it.

[What kind of simple things you can say in two dimensions?]

Let me do a general discussion. For $d = 0$ we saw that non super was $C^\infty(X)$ and the super things were $\Omega^0_{closed}(X)$. From a topologist's point of view, any two functions are concordant, whereas not any two of the closed zero forms are concordant. It has topological information. This is more pronounced if you look at the Euclidean case, then you get concordance classes the homology.

For $d = 1|0$ we have vector bundles with connections whereas in the super case you have vector bundles with flat Quillen connections ($\mathbb{Z}_2$ graded)

Over X it's less pronounced, the difference. If you look at just field theories, the space of field theories for $d = 1|0$ is contractible, and for $1|1$ it is not.

For $d = 2$ you want to look at partition functions for Euclidean field theories. Partition functions are integral modular forms in the super case. In the general case they are not holomorphic.