AMS Sectional Meeting
October 28, 2006

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October 29, 2006

1 The A_∞ Matrad and the Polytopes KK Ron Umble

[Slide talk]

Thanks for the invitation. The connection with this talk and some of the others is only via the associathedra. I want to specify the combinatorics of $KK_{n,m}$, and the dimension will be $n+m-3$. When n or m is one, we'll get Stasheff's associahedron. So matrad (they generalize operads) are double corollas θ^n_m, with data flowing upward. We start with Markl's fraction product, where monomials have the form

$$\alpha^y_x = \frac{\alpha^{y_1}_p \cdots \alpha^{y_q}_p}{\alpha^q_{x_1} \cdots \alpha^q_{x_p}}.$$

We'll generalize the fraction product to the matrad product. Let me explain it in dimensions up to three. The lower and upper leag sequences of a monomial are the x_i and y_j, respectively. We're also interested in the contact sequences, which are the outputs of the bottom and inputs of the top.

We'll need the $S-U$ diagonal of the associahedra. it's non-cocommutative and non-coassociative. We'll iterate it. View each component of $\Delta^{(k)}_K$ of a top dimensional face as a subcomplex of $K_p^{\times k+1}$. A nonvanishing matrad monomial in dimension up to three, well, the lower contact sequence of the $a^q_{x_i}$, the q decomposes into how the outputs line up, and that sequence will agree with some components of the iterated diagonal of up-rooted trees, and similarly for the upper contact sequence for down-rooted trees.

So, what's the point? It's A_∞ bialgebras. I was working with deformations of Hopf algebras, and students give talks, so I stupidly made a conjecture that one should be able to deform a dg Hopf algebra into an A_∞ bialgebra. We now have the bialgebra matrad $\mathscr{H}_\infty$ realized by the chains on this. Just as with up-rooted trees generating the A_∞ algebra operad, these corollas realize the A_∞ bialgebras. The endomorphisms on TH are canonically a matrad, so a map of matrads $\mathscr{H}_\infty \to End(TH)$ imposes an A_∞ bialgebra structure on H.

We're used to taking a sum of operations and squaring them to zero. So I want to define something similar. We need a biderivative which is a non-bilinear operation–

$$⊚\colon \mathrm{M} \times \mathrm{M} \to \mathrm{M} \times \mathrm{M} \to \mathrm{M} \to M$$

So we'll get $dS\omega$ as follows. We start with $d = \omega_1^1$, and extend it linearly across $(H^{\otimes p})^{\otimes q}$. Then we extend $\sum_{j\geq 1} \omega_1^j : H \to T^\circ H$ as a derivation, and cofreely extend as a coderivation horizontally. Then extend as a Δp derivation for each i, and a Δp coderivation.

2 Symmetric Brace algebras Tom Lada

All right, well, because this morning session involves A_∞ and A_∞ algebras. I'll show you what this has to do with these.

Brace algebras are a collection of multilinear braces $x\{x_1, \ldots, x_n\}$ that satisfy $x\{\} = x$ and

$$x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\} = \sum \epsilon x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_m}, x_m\{\ldots\}\ldots\}$$

We'll have symmetric braces, but they wil lbe graded symmetric in the variables, so that the defining relations where the sum is taken over unshuffle sequences.

Let's look at a couple of examples.

The nonsymmetric brace relation is

$$x\{x_1, x_2\}\{y\} = x\{y, x_1, x_2\} + x\{x_1, y, x_2\} + x\{x_1, x_2, y\} + x\{x_1\{y\}, x_2\} + x\{x_1, x_2\{y\}\}.$$

The symmetric version is

$$x\langle x_1, x_2\rangle\langle y\rangle = x\langle x_1, x_2, y\rangle + x\langle x_1\langle y\rangle, x_2\rangle + x\langle x_1, x_2\langle y\rangle\rangle.$$

These are motivated by the following. Let V be a graded vector space, and consider

$$B_s(V) = \bigoplus_{p-k+1=s} Hom(V^{\otimes x}, V)_p$$

where these are k-multilinear maps of degree p.

A nice example is the following. Consider B_{-1}, and suppose we have a collection μ_k of degree $k-2$ in there. Let μ be the formal sum of the μ_i, then an A_∞ algebra on V is defined by the brace relation $\mu\{\mu\} = 0$. Ignoring signs

$$\mu\{\mu\}(x, y, z) = \mu_1\{\mu_3\}(x, y, z) + \mu_2\{\mu_2\}(x, y, z),$$

which once expanded out says that the μ_2 is homotopy associative.

Now if you symmetrize, paying attention only to the alternating or antisymmetric maps, we can define this for a symmetric brace $\langle\ \rangle$. A fundamental example in this situation is that if you restrict to l_k of degree $k-2$ in $B_{-1}(V)$ then the relation $l\langle l\rangle = 0$ gives an L_∞ structure on V for $l = \sum l_k$.

We can look at these brace algebras, symmetric and non-symmetric, you can do some symmetrization. If you have a brace algebra structure on U then you can symmetrize it to a symmetric brace algebra on U. Then if you take special examples, these Hom sets, we can skew symmetrize maps, so that skew symmetrization commutes with braces. So skew symmetrization will be an algebra homomorphism from the brace to the symmetric brace structure.

As a corollary, the antisymmetrization of an A_∞ algebra structure μ yields an L_∞ algebra structure. Given $\mu\{\mu\} = 0$, we have $0 = as(\mu\{\mu\}) = as(\mu)\langle as(\mu)\rangle = l\langle l\rangle$.

Why did we do this to these ∞ algebras? It was motivated by a physics problem. Let me look at this particular, say I have a small graded vector space Ξ and Φ in grading 0 and -1. We want to construct a brackt on $Hom(L^{\otimes p}, L)^{as}$ so that the existence of an L_∞ structure on L is equivalent to this bracket satisfying the Jacobi identity.

We do this by constructing to maps ∇ and υ that come from physics, and defining $l = \nabla + \upsilon$, and then doing some lengthy calculations.

3 Michael Penkava
Miniversal deformations and the moduli space of isomorphism classes of algebras

I'm not sure exactly what the title is, my titles are so close together I never know which is the right one.

I want to discuss how to construct the moduli space of a species of algebra of a certain dimension, and stratify it by very nice orbifolds, with th eonly connections between the strata being jump deformations.

If you look at Lie algebras on a three dimensional space, the relations are quadratic. So it's some algebraic variety. So then you have to let a group act on this because of equivalence. All of the non-Hausdorff behaviour, under this group action, though, is collected in jump deformations.

Let me give you some motivation in a complicated sense.

Here's a picture of the moduli space of four-dimensional Lie algebras. Let me give you an easier example, the moduli space of three dimensional Lie algebras. There is one family of these and then three special points. Let me give you the structure. I can represent the structure as a matrix, a three by three matrix $\begin{array}{c|ccc} & 12 & 13 & 23 \\ 1 & & & \\ 2 & & & \\ 3 & & & \end{array}$. So classical theory says that there is only one semisimple one, with nonzero determinant. The others have $0 \to W' \to W \to \mathbb{C} \to 0$. So we get same smaller matrix $\begin{bmatrix} A' & \delta \\ 0 & 0\,0 \end{bmatrix}$. So these turn out to be the same when $\delta \sim \delta'$ up to constant multiples. The possibilities are

$$d_{\lambda,\mu} = \begin{bmatrix} \lambda & 1 \\ & \mu \end{bmatrix}, d_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, d_1 = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$$

The $d_{\lambda,\mu}$ is projective, and there is an action of the symmetric group, taking $d_{\lambda,\mu}$ to $d_{\mu,\lambda}$. So we get $\mathbb{P}^1/\Sigma_2$. Now we can ask ourself what the orbifold points are. We look for something where the stabilizer of the point is nontrivial. They are $(1, 1)$ and $(1, -1)$. The second is stabilized because you can multiply by constant multiples. So we might expect some unusual things to happen at these points, and they do. So I have a table of the cohomology of three-dimensional algebras.

Type	H^1	H^2	H^3
d_1	4	5	2
d_2	3	3	0
$d_{1,1}$	1	1	0
$d_{\lambda,\mu}$	1	1	0
$d_{1,0}$	2	1	0
$d_{1,-1}$	1	2	1
$d_3 = \mathfrak{sl}_2(\mathbb{C})$	0	0	0

So you can se that $d_{\lambda,\mu}$ is the same as $r_3, \lambda/\mu$ unless $\mu = \lambda$, where r_3 is the diagonal matrix with μ and λ. When I deform these, however, with a t, I can make a jump deformation from the identity to the Jordan block. However, moving from the Jordan block, I can only move along the family. Our first idea when we realized this was, kick out the one that they include in the family. You get a better picture when you look at the deformations of an algebra.

Now I'll need a little bit more board space.

Let's start by describing what these kind of algebras are. Let me give you a definition that might justify, I wasn't using Lie algebras from the exterior algebra but the symmetric.

Let V be a $\mathbb{Z}_2$-graded vector space. A Lie algebra on V is usually described as a map $V \wedge V \to V$. This is beautiful classically, but you run into sign problems taking this to the L_∞ case. You can still give a good bracket. I realized, it turns out that if you let W be the parity reversion of V, where $W_o = V_e$, and $W_e = V_o$. People who like $\mathbb{Z}$ grading either suspends or desuspends. No one can remember which of degree ± 1 goes with which of (de)suspension.

So I want to create some structures on this. Let $C(W)$ be exactly the coderivations of $T(W)$, the tensor algebra on W. This is the same thing as $Hom(T(W), W)$, and also equal to $\prod_0^\infty C^n(W)$, where $C^n(W) = Hom(T^n(W), W)$. I used to write, this is the nth tensor power of W. People disagree whether T^n is the tensor power or the symmetric power.

The important thing is that $C(W)$ is a graded Lie algebra, and we can say $\varphi \in C(W)$ has the form $\varphi = \varphi_0 + \varphi_1 + \dots$ I don't like to look at this sum, but sometimes it's convenient.

Okay, so let d be an odd coderivation such that $d = d_1 + d_2 + \dots$, then d determines an A_∞ algebra sturcture on V if $[d, d] = 0$. Similarly, if $C(W)$ is taken to be the coderivations of $S(W)$, then $[d, d] = 0$ determines an L_∞ algebra on V.

I'm going to gloss over that I'm interested in the symmetric and tensor coalgebra, not the algebra.

What's interesting about this is that henceforth everything in the theory is identical. These algebras come equipped with their cohomology. I mean the cohomology that deformation theory cares about. Define $D : C(W) \to C(W)$ by $D(\varphi) = [d, \varphi]$, so that $D^2 = 0$. This comes from $[d, d] = 0$ and d odd. With $\mathbb{Z}$ grading we have d of degree one or minus one instead of just odd.

That's the definition. What's going to determine, suppose g is an automorphism of $T(W)$ or maybe $S(W)$. Then we say that $d' \sim d$ if $d' = g * (d) = g^{-1}dg$ for some g.

Now it's interesting to note that automorphisms of the tensor algebra are more general than those of the vector space. Now, automorphisms of the vector space give you automorphisms of the tensor alegbra, which I call linear and which are called in the literature strict.

Now let me tell you what an associative algebra is, or a Lie algebra. This is given by $d = d_2$. We used to call this quadratic, but the operad is quadratic regardless, so we should call it binary rather than quadratic. Now, you might not like this notion of associativity either, because it's not precisely associativity or Jacobi on W, but they are when translated back to V. We now need to understand how to deform these algebras. Here's a classical deformation. Let $d_t = d + \varphi t$. We need $[d_t, d_t] = 0 \mod t^2$. That means that $D(\varphi) = 0$. So there is some relation between cohomology and infinitessimal deformations.

Now I'll talk about formal deformations. These are $d_t = d + \varphi_i t^i$, with the same requirement, that $[d_t, d_t] = 0$. Now the difficulty in simply using these ideas is that the theory of defining these is very complicated, involving Massey products. We constructed a versal deformation, a formal deformation with as many parameters as the dimension of the homology. Let $\langle \delta^i \rangle$ be a basis for $H_o^2(d)$. Then if I let $d^{inf} = d + \delta^i t_i$. This is a universal infinitessimal deformation. From this universal infinitessimal deformation. Then modulo second order terms this is zero, but without that modulus, it is not zero. In fact, $[d^{inf}, d^{inf}]$ looks like $\alpha^k a_k^{ij} t_i t_j + \beta^k b_k^{ij} t_i t_j$, where $\langle \alpha^k \rangle$ is a (pre)basis of $H_e^3(d)$, and $\langle \beta^k \rangle$ is a (pre)basis of $B_e^3(d)$. Here prebasis means that they are representatives, or representatives of what they are boundaries of. So the β^k will be odd and the α^k even. I can get rid of the β terms, so like a normal algebraist, I continue forever, getting rid of the boundaries, and then declare that whatever is left on the right is equal to zero, meaning you have to set some polynamial in the t_i equal to zero. I now can take a Lie algebra, put it into a program, and say "Versal d," and out pops the versal deformation, the relations, and I can study what the deformations look like nearby. If I take things that satisfy this, I can determine all the things that are close. Out of solving the formal deformations, I find all the convergent ones. The amazing thing is that the relations, which are formal power series, are always rational functions of the parameters. I have to look, in the neighborhood of a point, at jump deformations, which look like $d_t = d_{\varphi_i} t^i$, where for $t = 0$ this is isomorphic to the original, and then for all other values of t you get something different. By isolating what doesn't really belong in the neighborhood because of the jump, we can tell you what the neighborhoods are, and then look at things, and it turned out that for Lie algebras of dimension three and four, associative algebras of dimension 2 and $(1,2)$, real Lie algebras (where it's a little different) and L_∞ algebras of dimensions $(1,2)$ and $(2,1)$, and $\mathbb{Z}$-graded L_∞ algebras, all the pieces looked like $\mathbb{P}^n/\Sigma_{n+1}$, $\mathbb{P}^n$, and isolated points. These are compact orbifolds, projective even, and we've been proving pieces to the effect that the moduli space should always have such a stratification.

Thanks.

4 Marylin Daily