AMS Sectional Meeting
October 28, 2006

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A couple of announcements before we get started, we've had some cancellations on Sunday. I think I've met everybody in the room, but don't be surprised if I forget your name before this afternoon. I'll give it another minute. I guess we can get started.

1 A compact set of compactified moduli spaces Jim Stasheff

[Slide talk.]

This is a survey of various moduli spaces, their various compactifications, and their uses.

The original idea of a moduli space is an idea of having a parameterization of a family of structures. One of the main ideas is to have a continuous family of something or another, as opposed to, say, fiber bundles, where you get a discrete set.

The most classical moduli space is $\mathscr{M}_g$ the space of genus g Riemann surfaces. For g greater than zero, the set of equivalence classes, remarkably, the moduli space is a complex algebraic manifold. You can draw nice pictures for the torus but as this is a survey talk, I won't bother. For algebraic geometers this is very nice, but it doesn't always happen. Then you wave your hand and call it a stack and everything is okay.

Then that leads to n-tuples of points in a space. Look at a simple case. Suppose we look at distinct points on the real line modulo affine transformations. You can fix two of the points to be 0 and 1, so you break it up into things that sum up to one, so an open simplex. We could compactify to the closed simplex, but we want to keep track of how things approach collision. As you approach the end you have only one point moving. If you had two interior points, they could be approaching each other before they approached the end. So we could try to keep track of this as we compactify. So here with thanks to others, who drew this picture, are some pictures. If we go to the case of four interior points, we can see this information.

We've compactified the open simplex for four interior points as a pentagon. In general you get the associahedron, this was a few decades after I introduced the associahedron.

Another way to picture this is as a bubble tree, if the background is a circle. Take a simple example. Two points are coming close together, so instead of colliding they form a new circle bubbling off the previous one. Back to our original example, where the background is a complex manifold, to say that they are distinct points means we cut the fat diagonal out of the n-fold product of the sphere, in genus zero. In the compactification we look at stable complex curves. They can have nodal points, each line contains at least three special points (punctures and nodal points), and remarkably again we get a projective variety. One way of thinking what we get, as the structure changes we can pinch off so thot we get two spheres connected at a node.

My title and abstract were supposed to indicate that there are a variety of such moduli spaces. My notation is not standard. We've just talked about $\mathscr{M}_{g,p}$ and $\bar{M}m_{g,p}$, but we could also talk about $\mathscr{M}_{g,b,p,q}$, where b is the boundary and q are points marked on the boundary.

One way to picture the compactification is with blowing up. We have intersections, two at a time is no problem, but where there are triple or more, you cut out a neighborhood. There are multiple versions. You could leave this as a manifold with corners, or identify with the antipodal map.

Another approach to compactification si known as geometrification theory, where you add more information, as in the litle disks operad. That doesn't give you a compactification, but it gives you an operation that leads to an operad. Instead of thinking of those little disks, you can think of local coordinates centered at each of those points, and when you do that attaching you can sew them together, keeping the complex structure.

Now, once you start thinking this way, you can do other things in the same spirit. You can multiply examples. You can take moduli spaces of rational maps or of G-bundles, or whatever. The geometry can be differential or algebraic. Venturing into the algebraic, these are intimately related to deformation theory. Indeed, deformations of an algebra or other structure are considered up to isomorphism. The set of such isomorphism classes is the related moduli "space." An issue of importance is giving it a topology, homotopy type, or algebraic structure itself. It can be quite problematic.

To end this, let's talk about Vakil's Murphy's Law in algebraic geometry. The question is how bad can deformation spaces be. The answer is, as bad as possible unless you have some reason to think otherwise.

We have a ten minute break.

[Are there other families of polytopes that appear?]

Yes. Those pictures I drew as bubble trees could be circles bubbling off circles, two-spheres bubbling off, or disks bubbling off. He does the same idea but for the disk. You need at least three marked points on each disk. This gives another way to picture the associahedron. Instead of thinking of them as dots, think that you can have a subdisk. So what's going on?

You start with a subdisk. You can pinch off some of the subdisk, or not. This is the picture for the multiplihedron. If you have a context where your object is topological, and you have two such objects, your map in between may only be up to homotopy. If they're both A_∞, between associahedra, you get a cylinder with some combinatorial data in between. This is a picture.

In general you get something like a manifold with corners, usually. You get one point where there are more than three edges incident on a vertex. So you have a pentagon on top and another one on the bottom. One of the hardest problems is what to do once you get above three dimensions. Some people have come up with a PL subdivision of the cube. Chris Woodward has come up with the idea of the moduli space of quilted disks, which gives this in its compactification.

2 Moduli Space Actions on the Hochschild cochains of a Frobenius algebra Ralph Kauffman

So you heard the title, this is a subject, the idea is simple, but if you want to make it work, it takes 100 pages, that's bad. I hope you go out of this and say, this is clear. I'll give you one tool, the arc operad.

What are the motivations? Why should such moduli space actions exist? Deligne's conjecture says that you have an action by the chains of the little disks operad on the Hochschild cochains of a Frobenius algebra. This is a surface, so why not all surfaces?

Next, a motivation that is highbrow. Closed strings should give deformations of open strings. So if you have one D-brane, this is an algebra and the deformations are governed by HH^*, so a Riemann surface should give a correlator for closed strings, i.e., on HH^*

Lastly, in Chas Sullivan, if you have a surface in something, you should get operations on the loop space, the boundary is loops, marked as input and output. I should get a PROP.

A litle physics stream of consciousness. The little disks aree surfaces with boundary, TFT is connected to Frobenius algebras, there is a BV structure, there are D-branes. But little disks are flat (switch to cacti so you lose this), TFTs are cyclic, but you can add cyclic with no problem, do you deal with punctures or marked boundary (a boundary is the same as a puncture with a tangent vector), and what about higher genus (but that's this talk.)

The remaining question is how to go to the boundary, or what about Gromov Witten invariants?

This is something to think about. The history of the mathematics of string theory, people liked $\mathscr{M}_{g,n}$. I'm very old fashioned and look at the open moduli space, adding the vectors. If I play games like this I could have a nice machine to relate these two things. Going to the boundary should be related to GW-invariants. All roads lead to Rome, where Rome is the moduli space of curves.

Here's poor man's string theory, and you'll get the arc operad. Two strings can join and you get a surface with boundary, but you can also join together two arcs and vanish them. So we can put lenth paramaters on the S^1, and you trace out where the lengths go. You keep track of arcs with labels. This gives a partially measured foliation. If both are length one, they combine in one way, if you have lengths one-half you get another.

You can gather together all these objects into one space. I construct the space as a CW-complex, then the elements are exactly these. I have a surface with labelled boundary, arcs between the boundary, and so I have graphs on surfaces with a partial order by throwing out edges. Then I can associate with each of these graphs a simplex. I don't want parallel or intersecting arcs. I get a big cell-complex, with the action of the mapping class group of the surface. Now under the action, the quotient, some simplices have faces identified, others are completely identified. Look at the annulus, with one line, there is only one way. With two, I can put one arc straight. But then forgetting that one I can unwrap the other to be straight, so you identify the ends of the simplex (line) to get S^1, which you should think of as the BV.

Okay. So let $Arc_g^s(n)$ be subspaces of $A_{g,n+1}^s$ which consist of families hitting all boundary components, and $Arc(n)$ as the disjoint union over g and s. So fixing two elements, scale so that i and 0 agree, glue, and then cut and glue the bands according to their least common partition. So when two things don't match up you continue the seams from each side onto both.

This gives you the operad structure. The theorem is that the gluing gives you an operad which is cyclic, which is not surprising. You have lots of bands, you glue them, and you get a very complex thing. I have the bands with widths, the dotted and dashed lines are the cuts, and when I start cutting up here I come down over here, and so on. The geometry tells you you're cutting foliations so that whatever happens, you're good.

This is a large gadget containing everything I said before. Here are some suboperads.

I could just cut along the arcs, so that every complementary region is a polygon or a punctured polygon. So that is $Arc_{\#}$, and then I could get rid of punctures to $Arc_{\#}^0$. You could go to $GTree$ where the arcs run from 0 to i. Then you could pass to $Tree$ where you add $g = s = 0$, and $LinTree$, adding compatibility of the linear olders on the arcs. These are homotopy equivalent to Penner's $M_{g,n}^{dec\ \ s}$, exactly $M_{g,n}^n$, nothing, cacti, and spineless cacti.

Okay, how do you see this is a moduli space? You can take a dual graph construction. I can put a point in each region. Edges are common boundary. I get some sort of graph, which looks like a cactus. This here is a spineless cactus, if you know what that is. Otherwise, let them be defined as what I get via this dualization.

If I have puntures, I might get spikes sticking out. But, okay, without this I get a ribbon graph. If I have the lengths, I get the lengths of tangent vectors, and marked points on the boundaries, which gives the direction. So this is really a moduli space. What happens to the conformal structure under gluing? The action of the gluing is wild.

So theorems.

1. Deligne's conjecture (classic)
2. the same (cyclic)
3. the same (A_∞)
4. There is a rational operad on the collection of moduli spaces of genus g surfaces with marked points and tangent vectors. There is a problem because if the gluing matches up there are problems.
5. There is a chain operad induced by the above which acts on the Hochschild cochains of a Frobenius algebra extending the action of Deligne's conjecture
6. There is a quasi-PROP of Sullivan chord diagrams, which contains some of moduli space and some of Penner's compactification. Here I assume that arcs are running from in to out. I get some of moduli space if they are polygons, but once I glue I might move into the compactification.
7. There is a CW cell model for this prop whose cellular chains act in a dg $PROP$ fashion on the Hochschild cochains of a Frobenius algebra extending this action

So how do I do the moduli space action? I'll tell you how a surface with $\mathbb{Z}/\mathbb{Z}_2$ marked angles acts. I will define the operation if A is Frobenius, maps in $A^{*\otimes n} \otimes A$, but I can freely dualize since I'm Frobenius, os I get $A^{\otimes n+1}$ so the identifications are dg. The action here is to take the copies and map them to $A^{\otimes n+1}$. I have to give you elements in $A^{*\otimes |n|+|m|}$. You take your surface with arcs and $k+l$ boundaries, and duplicate the arcs on the i boundary, and the new angles are all decorated by 1, and if this is not possible the operation is zero. Then you decorate the angles marked by 1 by elements of A.

[I'm lost.]

There is a statement that might be made like "String topology is Kozsul dual to Gromov Witten"