Math Physics
December 14, 2006

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January 12, 2007

This talk may be somewhat abstruse. I will try to make it nice to the audience. One way is by systematically working out examples.

There is a symplectic invariant called symplectic homology. I am not going to tell you what kind of manifolds it's defined on. Let X be an affine variety with a trivial canonical class K_X. We equip it with a symplectic form $\omega = -dd'h$ Then $SH_*(X)$ is a $= zZ$-graded $\mathbb{C}$-vector space often infinite dimensional in each degree.

As examples

1. $SH_*(\mathbb{C}^n) = 0$
2. $SH_*(X \times cC) = 0$
3. $SH_*((\mathbb{C}^*)^n = H^{-*}(\mathscr{L}T^n)$.

This is very good at distinguishing things.

One of the things I'm concerned about is flow of information in symplectic topology. In principle we have an infinite amount of output information from finitely much input information. I like to represent varieties not by gluing together charts but as Lefschetz fibrations.

So take a nice map $\pi : X \to \mathbb{C}$ an affine Lefschetz fibration, nondegenerate critical points and reasonably well-behaved at ∞. Choose a smooth fiber Y. Then for every critical point of π we get a vanishing cycle (Lagrangian) in Y called $(L_1, \dots, L_m)$.

Take $X = (C^*)^2$ and take $\pi(x, y) = x + y + \bar{x}\bar{y}$. This appears in physics as the mirror of $\mathbb{CP}^2$ but we don't need to know that. There are three critical values $\zeta^3 = 27$ and the fiber is a three-punctured torus. If you know the story you know I'm lying a little bit. From this information you can reconstruct the total space. I'm going to take this fiber information and extract certain algebraic data. I will keep the general definition very sketchy. There are two things B and A, and A is a subset of B. So B is the part of the Fukaya category $\mathscr{F}(Y)$ with objects (L_i) where $Hom_B(L_i, L_j)$, if they intersect transversally there is a Hom for each intersection point. The interesting thing is the composition. For A you just throw away part af the data. If $i < j$ you keep the whole thing, if $i > j$ you get rid of everything, and if $i = j$ then you get just the identity.

This is kind of obscure. You might think that throwing away the information to get from B to A it doesn't really matter, there is a loss of information in the composition.

In the example, we have a with L_1, L_2, and L_3 each pair of which has three intersection points. So you get

$$\begin{tikzcd} L_1 \arrow[r, "V"] \arrow[rr, bend right, "\wedge^2 V"'] & L^2 \arrow[r, "V"] & L^3 \end{tikzcd}$$

with the composition give by the wedge.

What about B? You have a bunch of morphisms back and nontrivial morphisms L_i to itself. As an algebra this is $\wedge^*(V) \rtimes \mathbb{Z}/3$. You could have taken A and doubled it back up to obtain this. But that's not quite true. B has higher order products (is not formal). So $\mu^3(v_1, v_2, v_3) = v_1 v_2 v_3 \in \mathbb{C}1_{L_1}$. I didn't explain the $Hom(L_i, L_i)$. I move it slightly to get a transverse copy of it. Then you find just one thing with the order L_1, L_2, L_3, L_1'. You could say, what happens if I move something a little bit? You will always get one quadrilateral.

In this particular case we have homological mirror symmetry, $D(A) = H^*(\text{manifold})$ and get the catergory of finite dimensional A-modules, diffrential graded. Then $Y^* =$[unintelligible]

So there's a basic understanding of these two things. My question is, how are these two objects actually related, and now it will start getting very algebraic. What is the fundamental exact sequence?

$$0 \to A \to B \to B/A \to 0$$

This is a short exact sequence of A-modules. It helps to just think of these as algebras. Here $B/A = A^\vee[-\deg\ Y]$. Since we have the short exact sequence we get the boundary homomorphism $B/A \to A$ of degree one.

It's hard to understand the information in a boundary homomorphism. It measures how far this is from being split. You want to think of bimodules as acting on modules. A bimodule is useful because it supplies you with a functor $\phi_P = \otimes P$. If $P = A$ then $\phi_P = id$. If $P = A^\vee$ then $\phi_P = S$.

Here $S = K_{X^*} \otimes \bullet[2]$. So δ is a section of $K_{X^*}^{-1}$. This section is precisely the definition of a section of Y^*. Apparently this has nothing to do with symplectic homology. But you'll have to bear with me for one more step and then things will slowly start to converge. You have A and B. I want to put in another object D which is a curved A_∞-category. In passage to D you should only get information about the total space. The objects are the same as what we had before, $L_1, \ldots, L_m$. Suppose I take $Hom_B(L_i, L_j)[[t]]$ where the degree of t is two and the leading term is in $Hom_A(L_i, L_j)$. The structure maps it has are almost all inherited from B itself. You just need $\mu_D^2 = t1_{L_1} \in Hom_D(L_i, L_i)^2$. So going backwards you have to put in at least one t.

The conjecture then is

Conjecture 1 *The Hochschild homology* $HH_*(D,D)$ *is* $SH_*(X)$.

Here all of D should be an invariant of the total space, but we need to say up to what. What do we know about this? Well, this is an interesting thing to check. There are a number of ways to represent this as a total space of a Lefschetz fibration. There are cases where you know this and might want to check it. So there's a theorem that says you cannot calculate $SH_*(X)$ algebraically in finite time.

I'm interested in this question of how information in symplectic geometry is related to itself. I want to use some information about this, let's see. When you look at the chain complex that defines $HH_*(D,D)$, huge chunks are a cycle. Throw these out and what's left is $\overset{\lim}{\leftarrow}$ $HH_*(A, B/A[-\deg A])$. This means that this whole gadget only uses A and δ. Now what is the use of this? I will apply it to my example case. Then

$$HH_*(A, B/A[-\dim_{\mathbb{C}} X]) \cong Tor_{X^* \times X^*}(\mathscr{O}_\Delta, \mathscr{O}_\Delta \otimes R^{op}_{X^*})$$

So once you dualize this turns out to be $H^*(X^*, \Omega^*_X \otimes K^P_{X^*})$.

So this is $H^{2-*}(X^*, \wedge^* T_{X^*} \otimes K^{1-p}_{X^*}$. You're allowing sections with poles of higher order along the advisor. In the end what happens is that the Hochschild homology of $\mathscr{D}$ turns out to be $\Gamma(\mathscr{U}^*, \wedge^*(T_{\mathscr{U}}$ where $\mathscr{U}^* = X^* \backslash Y^*$.

So $SH_*(X) \cong H^{-*}(\mathscr{L}T^2) = \prod H^{-*}(T^2)$. In general you would want to do this backwards. I'm basically done, let me say a couple of words about what I should do. This carries a bunch of the structure that the closed string theory holds. When you look at the full structure, it may use more than the structure of A.

Thank you very much.

[Usually Hochschild theory has circle actions.]

The equivariant symplectic homology should be the [unintelligible]. Conjecturally it's all the same. If you look at how it's defined in terms of Riemann surfaces, there's a circle action I haven't used.

[Russian question?]

So, yes yes yes of course. Symplectic homology for the total space is the direct limit of the fixed point Floer homologies for the boundary map. Mirror symmetry is easy to prove for [unintelligible]varieties. This information is actually contained in the fact that the total space of the fibration is a torus. You can make random constructions by randomly changing vanishing cycles.

[Let me go back. Don't do the homology. On the space maybe there's a circle action there?]

These both come with the structure of a mixed complex. The obvious idea is that those should be the same structure.

[Is there a mirror [unintelligible]?]

You have to define modules over D correctly but it should be sheaves over D^*.

[Is there a version of topological mirror symmetry in terms of [unintelligible]with the symplectic homology.

[unintelligible].