Mathematical Physics
September 6, 2006

Gabriel C. Drummond-Cole

September 14, 2006

[Course overview.]

Let's get started with Newtonian mechanics. The prerequisite is that you know what a manifold is. I won't assume Riemannian geometry. I'm trying to keep things simple, so things will be on flat affine spaces. If you want to sup it up in your head as we go along, feel free.

So classical mechanics, we're talking about the physics (and math) of a single particle moving in some Euclidean space. So if you want to play along at home it could be a Riemannian manifold. The mathematical models are paths x, maps from time $M^1 \to X$ where X is a Euclidean target space. (or possibly a Riemannian manifold). Do I need to define a Euclidean space? Sorry, I guess that's a little bit insulting.

All right, so there are two spaces that are going to color our approach to this, M^1 and X. Let's look at the structure of both of these spaces.

Okay, so time has physical significance. We attribute certain mathematical structure to it to correspond to this. In particular,

1. It's affine, meaning that it's not necessarily a vector space. How do you add two instances in time? You can't. You can talk about how much time has passed, so it's an affine space over a one dimensional vector space $T \cong \mathbb{R}$.

2. There are units, like seconds or hours. What sort of mathematical structure would units be associated with? A norm, a metric. In particular we have a translation invariant metric on affine time. In other words, T has an inner product on it.

3. We could potentially also attribute to it an orientation, a differentiation between going forward and backward. We'll hold off on that for now.

To have a cogent discussion, we want to do math, so I want to fix some affine coordinate $t : M^1 \to \mathbb{R}$. We want to choose this so that $|dt| = 1$.

The structure, excluding the third, gives us a symmetry group that we will talk about again and again. The symmetry group is the Euclidean group for M^1, which includes translations and reflections. So there is a short exact sequence $1 \to \text{Translations} \to \text{Euc}(M^1) \to O(T) \to 1$. That's the structure we associate to the domain.

Now what structure do we have in the range?

1. X is an affine space over a vector space V.
2. There is a (translation invariant) metric so that V is an inner product space. This measures distance on $\mathbb{E}^d$.

If you're playing along with Riemannian geometry, these are the conditions that

1. X is a smooth manifold
2. X has a Riemannian metric
3. The metric is complete so we can work globally on X.

Let's go back to the Euclidean space. Again there is an associated symmetry group $\text{Euc}(\mathbb{E}^d)$, where you have translations and then reflections and rotations. Again you have a short exact sequence

$$1 \to \underbrace{V}_{\text{translations}} \to \text{Euc}(\mathbb{E}^d) \to O(V) \to 1.$$

For a general Riemannian manifold the group of isometries will be smaller, meaning lower dimensional. Sometimes this group will even be trivial.

There is one last piece of data that we need to define classical mechanics on X. We have the model of maps from affine Euclidean time into a Riemannian manifold. The last piece of data is

- a potential energy function $\mathscr{V} : X \to \mathbb{R}$.
- A mass $n > 0$ of the particle.

A quick note on units. In this course we'll come across a few fundamental units. In classical mechanics we'll only come across mass, length, and time. Energy will have units mL^2/t^2.

Now we can finally define what classical mechanics is, or at least the classical mechanics of the system. So all togethher, for any target space X, all possible particle motions are modeled on paths $P = \text{Map}(M^1, X)$. We'll assume that the paths are smooth. Don't worry about putting a Frechet topology or whatever on this. Now given the potential energy function $\mathscr{V}$ the actual particle motions are paths x such that they satisfy Newton's second law

$$m\ddot{x}(t) = -\mathscr{V}'(x(t)) = -\text{grad}\mathscr{V}(x(t))$$

We're going to look at the space of solutions to this equation $\mathscr{M}$, the space of states. A solution is a state. It's also called a phase space. Let me mention some properties right off the bat.

- It will be clear soon that $\mathscr{M}$ is a smooth manifold, so we can do calculus on it.
- The affine Euclidean group for time $\mathrm{Euc}(M^1)$ acts on $\mathscr{M}$ on the right so in particular time translation acts on it. So $T_s(x)(t) = x(t-s)$.
- One other thing, the potential was a function of X. It can also be a function of time, so that the symmetry is broken. So $\mathrm{Euc}(M^1)$ no longer preserves the space of solutions.

Let me argue that this is a smooth manifold. Let's see this by breaking some symmetry. Choose an instant t_0 in time and by picking this we break the affine symmetry. Then there's a natural map $\mathscr{M} \to T\mathbb{E}^d = V \times \mathbb{E}^d$ given by $x \mapsto (\dot{x}(t), x(t))$. This is a bijection and you can just transfer the differentiable structure across.

Now we have a picture of what the space of solutions looks like. Let me give you some examples.

Example 1 *The free particle*
This is the case where it's moving in Euclidean space and $\mathscr{V} = 0$. *Then Newton's second law says* $m\ddot{x} = 0$. *Then* $\mathscr{M} = \{x(t) = p + vt | p \in \mathbb{E}^d, v \in V\}$. *So given a* t_0, *the map takes* $p + vt$ *to* (v, p). *In this case the map doesn't depend on* t.

Example 2 *This is a little less trivial but just as famous. It's the spring or harmonic oscillator.*
Implicitly you have to have a distinguished point where the spring is stable. I may as well take $X = \mathbb{R}^1$. *Then the potential energy is* $\mathscr{V} = \frac{1}{2}kx^2$, *where the units of* k *are* $\frac{M}{T^2}$.

Then Newton's second law says $m\ddot{x}(t) = -kx(t)$. *Then* $\mathscr{M} = \{p\cos(\sqrt{\frac{k}{m}}t) + \sqrt{\frac{m}{k}}v\sin(\sqrt{\frac{k}{m}}t) | p, v \in \mathbb{R}\}$. *So then* $p\cos(\sqrt{\frac{k}{m}}t) + \sqrt{\frac{m}{k}}v\sin(\sqrt{\frac{k}{m}}t) \overset{t=0}{\mapsto} (v, p)$.

There are other things I can point out about the space of solutions. We have the symmetry group of the domain that acts on the solutions. What about the symmetry group of the target? How does that naturally act on the space of solutions? it acts on the left, but only those isometries that preserve the potential. An isometry that changes the potential will not preserve the space of solutions. What's true about the two group actions? They commute. The time group will have to do with dynamics, the target group with kinematics. If $X = \mathbb{E}^d$, and $\mathscr{V} = 0$, so that we're talking about the free particle, then the entire Euclidean group acts on the space of solutions, since everything preserves the 0 potential. In particular, if A is an affine Euclidean transformation and its derivative is in $O(V)$, then $p + vt \mapsto Ap + (dA \cdot v)t$.

Let me wrap up what I've said today, which isn't much. In summary, we've discerned that the space of solutions $\mathscr{M}$ to Newton's second law has the following structure:

- There's a right action by the Euclidean time group $\mathrm{Euc}(M^1)$ acting by right composition with the inverse.
- There's a left action by potential-preserving isometries of X, $\mathrm{Isom}(X, \mathscr{V})$.
- For $t_0 \in M^1$ there's a natural diffeomorphism $\mathscr{M} \overset{t_0}{\cong} TX$.

Next time I hope to make this fit in with the idea of a symplectic structure.