Mathematical Physics
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Today we want to talk about Einstein's theory of special relativity (SR). This is a very simple theory in some sense because it is based on two principles.

1. Relativity (Galileo). There is no such thing as absolute velocity. $F = m\ddot{x}$ and the velocity is a constant of integration, always with respect to a particular frame. Then forces also have to shift it the correct manner. This is built into Newtonian mechanics. Galileo was riding on his horse throwing balls in the air. They still followed parabolic trajectories. It's hard, I guess, to ride a horse at constant velocity. He concluded that the mechanics were independent of the frame, a radical concept at that time.

   I'm using the word frame to mean approximately a coordinate system. In Galilean relativity there is the concept of an "inertial frame." Two frames are equivalent when they differ by a constant velocity. An inertial frame is equivalent to a rest frame.

   An inertial frame can be characterized by Newton's first law, that bodies in motion remain in motion at the same speed unless acted on by a force. Here are characterizations of an inertial frame.

   (a) The spatial distance between any two points is independent of time.
   (b) All hypothetical clocks are synchronized and running at the same rate.
   (c) The geometry of space at any constant time t ($dt = 0$) is Euclidean. This is special relativity, things will change in general relativity.

   So far we haven't really said anything new. In reference to math versus physics, special relativity doesn't have very much to it mathwise. The problem is that things are counterintuitive because of:

2. Universality of the speed of light (Einstein).
   Say $\mathscr{O}, \mathscr{O}'$ are inertial. Then v and v' differ by a constant. Do an experiment to measure the speed of light. This is hard. Galileo tried to do these experiments, but he realized his reaction time was too high. So in this approximation the speed of light is infinite.

   In this Einsteinian theory, no matter the relative velocities of $\mathscr{O}$ and $\mathscr{O}'$, the speed of light is the same number in both. So if you fire a gun out of the window of a car, you add the muzzle speed to the speed of the car. That doesn't work any more in special relativity.

Whenever something like this happens, we say that c is a fundamental constant. You can replace t with $x^0 = ct$. Then you get spatial coordinates that include c as a fundamental constant. This assumption means that there is no difference between space and time. This is sometimes called "God's units."

The approach I'm borrowing is to do everything geometrically. We'll replace space and time with space-time. Some people drop the hyphen. So space still has the coordinates x^i for $i \in \{1, 2, 3\}$, and now we have $x^0 = ct = t$. We didn't get to any actual solutions in Newtonian mechanics, but if we did we would draw diagrams. For example, for the oscillator we'd draw like a sinusoidal curve. We'll do the same thing now. Time will be vertical and space will be horizontal. If you had, for example, some point p, which, by the way, is called an event, it has a time and some coordinates. A particle is a vertical line, a static particle. That's called a world line.

A photon has velocity 1. This should be a diagonal line of slope 1. The world lines of light particles are at forty-five degree angles. We'll be comparing the frames of two inertial observers.

In these diagrams, often the light cone is drawn in. I should say, if I drew this in 2+1 dimensions, you would get a cone.

Now we want to pick two frames relative to one another. We want to talk about how things are different in the different frames. If $v = \frac{dx}{dt}$ then $\frac{1}{v}$ is the slope of the $\mathscr{O}'$ world line according to $\mathscr{O}$.

Let's use natural coordinates for $\mathscr{O}'$. He's sitting at $x = 0$. At time $t = 0$ a truck ran over me. I want a really close pass. The truck got closer and closer and then he gets further and further away. We found that this truck guy is going along a straight line trajectory with a certain slope that I can't quite get. He's choosing coordinates where he's at rest. Now in his coordinate system, he is at rest.

$t = 0$ is sometimes called a surface of simultaneity. We'll find that he will not agree with my concept of when two events happen simultaneously.

This is something so engrained in our thinking, whenever we give a paradox we're abusing simultaneity and special relativity. So let's talk about a special property, if $t' = 0$, go to some point $t' = a$ and equally far backward, $t' = -a$. Then the distance along the axis is the same in time. Well, if I looked at my light cone, suppose I had an object that gave a flash, the light would travel at 45 degree angles, so that the world line of the light goes out like this. Later on my clock I see they've travelled out a distance a. This is a special property of this surface. All points on the light front get there at the same time.

I'm using light to construct surfaces because it's the anly thing we both agree on.

Let me translate this picture, then. Take $-a$ in t' coordinates. These are now the coordinates according to $\mathscr{O}$. Then you know how the light bouncing like this works under the change of coordinates. So this shifts the axis. Then the shift takes the axis to a plane with relative slope v.

Next we have to calibrate the coordinates. How do the seconds tick off on a different world line. What do the distances look like when measured by $\mathscr{O}'$? In order to do all of this we have to define the invariant interval.

Now in one dimension we have $\frac{dx}{dt} = c$. But in multiple dimensions we want $\vec{v} \cdot \vec{v} = c^2$. No we can write this as $dx^i \delta_{ij} dx^j = cdt^2$. This translates into $-(dx^0)^2 + dx^i dx_i = 0$. We all agree that this is true, so $-(dx'^0)^2 + dx'^i dx'_i$ So define $ds^2 = \eta_{ab} dx^a dx^b$. This has $\eta_{00} = -1$ and $\eta_{ij} = \delta_{ij}$. It turns out that ds is invariant even when it isn't zero.