# Mathematical Physics
# September 19, 2006

Gabriel C. Drummond-Cole

September 19, 2006

All right, so any questions from class?

[This is a stupid question. We've been saying that having a second order ODE then having two initial conditions gives you a solution. Is this standard?]

Spivak has a spiel about this.

Today, well, we've talked about Lagrangian mechanics. Now we're going to talk about symmetries from the Lagrangian point of view. This will be symmetries redux. I like to pronounce that redux. I also like hors d'oevres and mercy buckets. I guess none of you laugh when Pepe Le Peu says "le mew?"

Okay, anyway, we have the Lagrangian, which is a map $L : (\mathscr{P} = \text{Maps}(M^1, X)) \to \text{Densities}(M^1)$. We have then $L(x) = \{\frac{1}{2}m|\dot{x}(t)|^2 - \mathscr{V}(x(t))\}|dt|$.

Now $L$ encodes

- The Euler Lagrange equations with solution space $\mathscr{M}$.
- A symplectic structure $\omega$ on $\mathscr{M}$.
- 1-forms on $\mathscr{M}$.

We associote a "Lagrangian" with a "theory." So the symmetries of the theory are the symmetries of the Lagrangian, which I will define in a minute. In other words, we should get that the symmetries of $L$ are symmetries of $(M, \omega)$. In the Hamiltonian picture, the other thing we should think about is that infinitessimal symmetries imply conserved charges. Remember that a charge was an observable, and if $\zeta$ was an infinitessimal symplectomorphism that [*unintelligible*], that would imply the existence of a conserved charge..

In the Lagrangian picture this becomes even more precise.

What is a symmetry? We're talking about infinitessimal symmetries. We want to work off-shell. Let $\xi$ be in $\mathscr{X}(\mathscr{P})$ be a vector field on the space of paths. In other words, if I have a path $x \in \mathscr{P}$, then $\xi_x \in T_x P = C^\infty(M^1, x^*TX)$. And really we only look at "local" vector fields, meaning that the value of $\xi_x(t)$ only depends on the $k$-jet of $x$. We'll see examples today. Those are the sorts of infinitessimal symmetries we'll be dealing with.

If I have such a vector field, to what extent does $\xi$ need to preserve $L$ so that it is an infinitessimal symmetry on $\mathscr{M}$?

Certainly it's true if $\text{Lie}(\xi)L = 0$? Then $\xi$ certianly preserves the Euler Lagrange equations, so $\mathscr{M}$.

It turns out that's too stringent to get everything, so let me point out that when we derived the Euler Lagrange equations, there was a caveat on the variations we considered. We only considered those with compact support. These were sections of the pullback bundle. In other words, we threw away the boundary terms when we did the equations, just like William did yesterday.

We throw away the boundary terms. So we don't require that it be preserved on the dot, but rather that the infinitessimal variation be exact, $\text{Lie}(\xi)L = \frac{d\alpha_\xi}{dt}|dt|$, where $\alpha_\xi : \mathscr{P} \to$ functions on $M^1$. Again $\alpha_\xi$ is local sa that's still preserved So $L$ can change by an exact term for $\xi$ to preserve the Euler Lagrange equations and therefore the space of solutions.

The idea is that the Euler Lagrange equations, I can say it in words or do a computation. If I differentiate the Lagrangian, say I've changed it infinitessimally. Now $L$ is infinitessimally $L + \frac{d\alpha}{dt}|dt|$. Now the action is the integral of this over $[t_i, t_f]$. The integral of the second piece disappears. This is under very special variations that this disappears and you get the Euler Lagrange equations. These two things come together and you get Voltron. There's a blazing sword somewhere.

The upshot is, all we really require is that $L$ can change up to an exact term.

So that's the idea, we have various sorts of symmetries. There are those that preserve $L$ on the nose and those that only preserve it up to an exact term. The idea now is to study these guys and see how they give preserved charges. That's Noether's theorem. I'm going to build this up slowly. I'll do it in cases. The first case will be manifest symmetries, where the Lagrangian are preserved on the nose, $\text{Lie}(\xi)L = 0$. Case two will be when things are only preserved up to an exact term. This is called a non-manifest symmetry. Case three will be a generalization of case one, which will remain mysterious for now. This is also called manifest symmetry.

Okay, so case one. This is manifest symmetry, type A. This is my nomenclature. I wouldn't try pulling that in a talk, especially around physicists.

Okay, so I have $0 = \text{Lie}(\xi)L = Lie(\xi)\{\frac{m}{2}|\dot{x}(t)|^2 - \mathscr{V}(x(t))\}|dt|$. We did this in class, and what you get is essentially the left hand side of the Euler Lagrange equations,

$$\{-\langle m\ddot{x}(t) + \text{grad}\mathscr{V}(x(t)), \xi_x(t)\rangle + \frac{d}{dt}\langle m\dot{x}(t), \xi_x(t)\rangle\}|dt|.$$

On-shell, $x \in \mathscr{M}$, that first term is zero. So if $x$ is a solution, I have $\frac{d}{dt}\langle m\dot{x}(t), \xi_x(t)\rangle = 0$. That looks like a conserved quantity to me. So look at this observable, $\mathscr{O}_\xi(x) = \langle m\dot{x}(t), \xi_x(t)\rangle$. This is called the Noether charge associated to $\xi$, and it is conserved. It's not conserved off-shell, only on-shell.

[Is that spiral a zero?]

It's the madness you descend into when you look into the heart of zero, really look into it.

Here's a nice example of a manifest symmetry. Assume $X = \mathbb{E}^d$ is Euclidean space and we're dealing with the free particle. Let $\{e_i\}$ be an orthonormal basis for the vector space over which $\mathbb{E}$ is an affine space, and let $\{x^i\}$ be compatible coordinates on $\mathbb{E}$.

Consider the following transformation on the space of paths. It's the transformation $x(t) \mapsto x(t) + e_i$.

The infinitessimal version of that is $\xi_x(t) = e_i$. All right, for the free particle, the Lagrangian is just the kinetic term, and I hope it's clear that $\mathrm{Lie}(\xi)\{\frac{m}{2}|\dot{x}(t)|^2\}|dt| = 0$.

Let's compute the Noether charge. That's $\langle m\dot{x}(t), e_i\rangle = m\dot{x}^i(t)$, and this is $p^i$, the $i$th component of momentum. This is independent of time. So momentum is our conserved Noether charge for this symmetry. So in other words, translational symmetry implies conservation of momentum, and vice versa.

**Exercise 1** *Hopefully it's clear that this is invariant under rotations, so you can compute infinitessimal rotations, and then compute the Noether charge. All you have to do is assume $\mathscr{V}$ is invariant under $O(V)$.*

Taking the symplectic gradient of momentum gives translational symmetry. For homework,

**Exercise 2** *Show that the symplectic gradient of momentum is translation on $\mathscr{M}$.*

[Does the theorem still hold in a general Riemannian manifold?]

It still holds but there are fewer symmetries generally.

[Is it clear that this is the same observable you get in the long exact sequence?]

No, and that's a good homework problem.

**Exercise 3** *Show that the Noether charge of $\xi$ obeys the equation that $\mathrm{grad}_\omega \mathscr{O}_\xi = \xi$. Remember the symplectic form in the Lagrangian setting is $\omega = \delta\gamma_t$. It's not too hard to show as long as you get all your ducks in a row.*

Let's go on to case two, non-manifest symmetry. I want to show that there's still a conserved charge here. Assume we have some function on path space $\alpha_\xi : \mathscr{P} \to \mathrm{functions}(M^1)$ such that $0 = \mathrm{Lie}(\xi)L - \frac{d\alpha_\xi}{dt}|dt|$. I do the same calculation as before. I get

$$\{-\langle m\ddot{x}(t) + \mathrm{grad}\mathscr{V}(x(t)), \xi_x(t)\rangle + \frac{d}{dt}\langle m\dot{x}(t), \xi_x(t)\rangle - \frac{d\alpha_\xi}{dt}\}|dt|$$

So it's clear what the conserved charge must be. If $x$ is on shell, I have the term from before, from the manifest case as well as the other term, $\frac{d}{dt}\{\langle m\dot{x}(t), \xi_x(t)\rangle - \alpha_\xi\} = 0$

So then $\mathscr{O}_\xi(x) = \langle m\dot{x}(t), \xi_x(t)\rangle - \alpha_\xi$ which we cal the conserved Noether charge for $\xi$ (with $\alpha$ given).

Let me give you an example of this. This is time translation as a non-manifest symmetry. So here $\xi_x$ is essentially $-\dot{x}$. To see that, parenthetical statement, recall that time translation acts by translation to the left, $T_s x(t) = x(t-s)$. So if I take the infinitessimal version of that and differentiate, $\frac{\partial}{\partial u}|_{u=0}(T_u x)(t) = -x(t)$. I need to show that this is a non-manifest symmetry. So

$$\mathrm{Lie}(\xi)L = \mathrm{Lie}(\xi)\{\frac{m}{2}|\dot{x}(t)|^2 - \mathscr{V}(x(t))\}|dt| = \{-\langle m\ddot{x}(t), -\dot{x}(t)\rangle - \langle \mathrm{grad}\mathscr{V}, -\dot{x}(t) + \frac{d}{dt}\langle m\dot{x}(t), -\dot{x}(t)\rangle\}|dt|.$$

This after a couple of steps of calculus is $\frac{d}{dt}\underbrace{(\frac{-m}{2}|\dot{x}|^2 + \mathscr{V}(x(t)) + C)}_{\alpha_\xi}$. This is two-fold, I killed two people with a rifle. So

$$\mathscr{O}_\xi = \langle m\dot{x}(t), -\dot{x}(t)\rangle - \alpha_\xi = \frac{-m}{2}|\dot{x}(t)|^2 - \mathscr{V}(x(t)) - C,$$

which is energy plus a constant, but I want the Noether charge to be spot on energy. Next time I'll show how to get to case B and make this a spot-on, manifest symmetry.