Mathematical Physics
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Okay, let's start. We're finally getting into Lagrangian mechanics. There's an announcement. There's an RTG seminar which has something to do with this class, the first one will be Monday. We'll both talk in it, but you're all invited. Since they're both RTG things, it would make sense to attend.

Last time we polished off Hamiltonian mechanics. Now we're going to go on to the Lagrangian picture. We have these solutions $\mathscr{M}$ to Newton's second law sitting inside the space of all paths, assumed smooth to keep everything simple. The idea was to derive this submanifold as a critical submanifold of a particular function called the action. That's what we're going to sort of embark on today.

So I said we're going to do that using the action but what is more basic, especially in mechanics, is the Lagrangian. More basically, we consider the Lagrangian density or, if you like, we can just call it the Lagrangian. This is a map $L : \mathscr{P} \to \text{Densities}(M^1)$, these being things we can integrate. So these are $f(t)dt$.

Back in the day you learned about top forms. If a manifold is orientable, you can integrate top forms, but if it's not orientable, you use densities instead.

[Dennis: I think it's important to keep the absolute value sign. The measure never goes to zero, but a form can.]

This whole thing is a symbol for a measure. It's a measure absolutely continuous with respect to the smooth measure and the density that relates these two is $f(t)$.

Okay, now why do I think this is more basic than the action? The idea is that while $L(x)$ is well defined for all $x \in \mathscr{P}$, often $S(x) = \int_{M^1} L(x) = \infty$. So it's often more useful to consider $L(x)$. When we do want to consider $S(x)$ we study finite length subsets of M^1. Despite that restriction we still achieve our submanifold. We still achieve the Euler-Lagrange equation.

You have this intuition that the way a particle motion will happen is by lowering the potential energy. How do you take something like that and construct an object whose minima give you the correct paths. That's what he's doing.

So for $L(x) = \{\frac{m}{2}|\dot{x}(t)|^2 - V(x(t))\}|dt|$, this is what we want to minimize over functions, not over t.

This is natural from the point of view of quantum mechanics, which you may not understand fully.

Note that $L(x)$ depends locally on t. This is tied very much to gluing properties. Something that wouldn't be local might include things that had t and another parameter τ, and you integrate over both dt and $d\tau$. Then that would depend at a given time on what happened at another time. Most theories we care about are local, depend only on the jet, nine out of ten depend only on the one-jet.

For finite time $t_0 < t_1$, define $S_{[t_0,t_1]}(x) = \int_{t_0}^{t_1} L(x) < \infty$.

We want to obtain the Euler-Lagrange equation by asking that S be stationary to first order under variations of x. In particular, we restrict to variations of x with compact support, so we can still work with the whole $S(x)$, which might not be well-defined, but will have well-defined variation. Choose our interval so that the compact support is contained in that interval.

That is the method we're going to use. Now I need to ask, what is a variation of a path? Before taking that variation, let's ask what it means. So what is a first order variation of x? We're asking, what is a tangent vector of $\mathscr{P}$ at x, i.e., elements of $T_x\mathscr{P}$. That's what we want to figure out. Let's start with x. Here's affine time M^1, and we map into our target space X. We end up with $x(t)$ a path in X. You can imagine, let's talk about, how do we normally talk about first order variations? What was one way of defining tangent vectors? Equivalence classes of curves. So we can look at curves in the space of paths that pass through x.

So we have a set of curves going through $x(t)$ parameterized by u. If we look at infinitessimal things coming off of x we can see how the path is being perturbed infinitessimally. So we can take $T_x\mathscr{P}$ to be $C^\infty(M^1, x^*TX)$, like vector fields along $x(t)$. This really has to be the pullback so that if the path crosses itself things work out correctly.

Let's compute the Euler-Lagrange equation for our Lagrangian density. Let x_u be a curve in $\mathscr{P}$ where $u \in (-\epsilon, \epsilon)$, with $x_0 = x$. This is a map $\bar{x} : (-\epsilon, \epsilon) \times M^1 \to X$. So let ζ be the infinitessimal vector field represented by the curve, so it is $\frac{\partial \bar{x}}{\partial u}|_{u=0} \in C^\infty(M^1, x^*TX)$.

For a moment ignore the support condition on ζ. Then let's also assume, to make things simple, that $X = \mathbb{E}^d$ and let $\langle, \ rangle$ be the inner product on V. Now we can compute.

All right, first of all, I'm asking that something be zero. I'm taking the variation with respect to the u parameter at $u = 0$. So that is

$$0 = \frac{d}{du}|_{u=0} S_{[t_0,t_1]}(x_u) = \frac{d}{du}|_{u=0} \int_{t_0}^{t_1} \{\frac{m}{2}|\dot{x}(t)|^2 - V(x(t))\}|dt|$$

$$= \int_{t_0}^{t_1} \{m\langle \dot{x}(t), \frac{\partial}{\partial u}\frac{\partial}{\partial t}\bar{x}(u,t)|_{u=0}\rangle - \langle \mathrm{grad} V(x(t)), \frac{\partial \bar{x}}{\partial u}|_{u=0}\rangle\}|dt|$$

If you go on to study more physics, you get to this point and you say, what now? You want to find the variation in terms of ζ. So this is equal to

$$\int_{t_0}^{t_1} \{m\langle \dot{x}(t), \frac{\partial}{\partial t}\zeta(t)\rangle - \langle \mathrm{grad} V(x(t)), \zeta(t)\rangle\}|dt|$$

We want everything linear in ζ so we change this to

$$\int_{t_0}^{t_1} -\langle m\ddot{x}(t) + \mathrm{grad} V(x(t)), \zeta(t)\rangle|dt| + m\langle \dot{x}(t), \zeta(t)\rangle|_{t_0}^{t_1}$$

Now using the support condition the boundary part of the equation goes away and so the integrand must be zero so that finally we get $m\ddot{x}(t) + \mathrm{grad} V(x(t)) = 0$, which is just Newton's second law or the Euler Lagrange equation for classical mechanics.

I claimed that this was going to give us our structure on the phase space, the symplectic structure and even the one-forms.

[You picked the right Lagrangian to get Newton's second law.]

Sure, it's rigged. I think this first came up in optics, that these kinds of things could be described from a variational principle.

When I come back to symmetry, there's a way that physicists come up with Lagrangians. Suppose that V were zero. Then we would have the Euclidean invariant kinetic energy. So you have a theory based on the one-jet, invariant under the Euclidean group. So you look for that, maybe with other conditions, a quadratic condition. Any other questions?

So let's try to derive the other structure on $\mathscr{M}$ from the Lagrangian. We'll work "on-shell," meaning, on the space of solutions to the Euler-Lagrange equation. We'll write "δ" for the exterior differential on $\mathscr{P}$ and also on the submanifold $\mathscr{M}$. This computation, the reason I didn't impose the support condition, is because I wanted the boundary term to pop up. For each $t \in M^1$, you can define $\gamma_t \in \Omega^1(\mathscr{M})$ defined by $\gamma_t(\zeta) = m\langle \dot{x}(t), \zeta(t)\rangle$. This is for $\zeta \in T_x\mathscr{P} = C^\infty(M^1, x^*TX)$. So for homework, show that for $\mathscr{M} \overset{\phi_{t_0}}{\cong} T^*X$ show that $\gamma_t = \phi_{t_0}^*\theta$. Do this for the special case $X = \mathbb{E}^d$.

So the Euler Lagrange equation implies that for any two moments in time, the difference of the two one-forms is exact. So $\delta S_{[t_0,t_1]} = \gamma_{t_1} - \gamma_{t_0}$ on $\mathscr{M}$. There are two conclusions to draw from this equation.

1. Define $\omega_t = \delta\gamma_t \in \Omega^2(\mathscr{M})$. Further, ω_t is independent of t on the space of solutions, because $\delta(\gamma_{t_1} - \gamma_{t_0}) = \delta^2 S_{[t_0,t_1]} = 0$.
2. The one-forms γ_t give us a (connection on) principle $\mathbb{R}$-bundle $R \to \mathscr{M}$ whose curvature is ω.

In conclusion, what have we done?

- The given data was a Lagrangian density $L : \mathscr{P} \to \text{Densities}(M^1)$. You started with that. It's like this piece of data, under the variational principle, it's like a zip file, all this stuff came out.
- By the variational principle we got
  1. The Euler Lagrange equation and hence our space of solutions
  2. our one-forms γ_t
  3. the symplectic structure ω.

  In the Hamiltonian setting things weren't canonical, or we had to guess that things were canonical. So the Lagrangian encodes all of that stuff, so it's in that sense that physicists equate a theory with a Lagrangian. All the information sits inside there waiting to be unzipped. When we talk about symmetries we'll see that in the Lagrangian context, there will be a canonical way to conserve charges. In the Hamiltonian context we just knew there was a conserved charge corresponding to infinitessimal symmetries preserving the Hamiltonian system.