Mathematical Physics
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Recall that for certain $\xi \in \mathscr{X}_\omega(\mathscr{M})$ (those for which the topological obstruction disappears) there exists a corresponding observable $\mathscr{O}_\xi \in C^\infty(\mathscr{M})$ such that $-d\mathscr{O}_\xi = \iota(\xi)_\omega$.

This describes the infinitessimal symmetry ξ via $\xi f = \{\mathscr{O}_\xi, f\}$ for any $f \in C^\infty(\mathscr{M})$.

We got as far as saying that there are a particular set of symmetries we're concerned with. There are the infinitessimal symmetries of time translation, $\zeta \in \mathscr{X}_\omega(\mathscr{M})$, and this has the observable $\mathscr{O}_\zeta$ where $-\mathscr{O}_\zeta$ is the energy or Hamiltonian. For a path x, $H(x) = \mathfrak{m}2|\dot{x}(t)|^2 + V(x(t))$. For $x \in \mathscr{M}$ this is independent of t.

[Is that obvious?]

Yes, I'll get to it in a second. That's where we left off last time. Any questions?

Before I finish off Hamiltonian dynamics, let me make some tangential but useful remarks about observables. Most of the observables we see in this class will be something like $\mathscr{O}_{(t,f)}$, defined for any time $t \in M^1$ and $f : X \to \mathbb{R}$. Then

$$\mathscr{O}_{(t,f)}(x) = f(x(t)).$$

Let me give another example, two examples.

1. If $X = \mathbb{E}^d$, then we can take $f = x^i$, and in this case $\mathscr{O}_{(t,f)}$ the x^i coordinate of the particle at time t.
2. The Hamiltonian, this is the energy of the particle at time t.

A jet of a function is essentially its Taylor series. The first type of observable depended on the 0-jet of the path; the Hamiltonian depends on the 1-jet. $\mathscr{O}_{(t,f)}$ is local in time, meaning it only depends on finitely many of these, only depends on a small neighborhood of a given time.

So what's the upshot? The structure on $\mathscr{M}$ is as follows. We have what is called a Hamiltonian system. That's the phase space with its symplectic structure and our function H, so the couple $(\mathscr{M}, H)$. So this is a symplectic manifold and a distinguished observable (energy) such that $-\xi_H$ is the infinitessimal time translation.

Let's look at the symmetries of this extended structure. Global symmetries are symplectomorphisms that preserve H. The infinitessimal symmetries are infintessimal symplectomorphisms $\xi \in \mathscr{X}(\mathscr{M})$ such that $\mathrm{Lie}(\xi)H = 0$. Now we're going to look at these symmetries in terms of observables.

So if Q is an observable that corresponds to an infinitessimal symmetry, then we have the following relation: $\{H, Q\} = 0$. Now, for any observable, never mind that it's a symmetry of any system, time translation flow on phase space is induced by H. So we get that $\dot{\mathscr{O}} = \{H, \mathscr{O}\}$.

Now we use the observable energy to tell us how things change with time. So now, thus, what we can conclude, assuming that the observable is a symmetry of the Hamiltonian system, for Q, if Q induces a symmetry of the Hamiltonian system, then we have a conservation law. We have that $\dot{Q} = \{H, Q\} = \mathrm{Lie}(-\xi_H)Q = \mathrm{Lie}(\xi_Q)H = 0$.

So look at $\dot{H}$. This is $\{H, H\}$ which is zero. So H is conserved. Such observables, here's more jargon, are called, and this is why I used Q, are called conserved charges.

So here's the big idea, big enough to put in a box. Symmetries imply conservation laws.

Exercise 1 *Compute these conserved charges. The physical situation is the free particle in Euclidean space. We have the huge symmetry group, which is the isometries of* $\mathbb{E}^d$.

Compute the conserved charges for translations and for rotations. These will be momentum and angular momentum.

Okay, let's talk about Lagrangian mechanics. For particles we have solutions to Newton's second law, $\mathscr{M} \subset \mathscr{P} = \mathrm{Map}(M^1, X)$. The idea of Lagrangian mechanics is to describe $\mathscr{M}$ as the critical submanifold of a function $S : \mathscr{P} \to \mathbb{R}$.

This function S is called the action, and $\mathscr{M}$ would be paths x such that $\delta S(x) = d_{\mathscr{P}} S(x) = 0$. So δ is the exterior derivative on $\mathscr{P}$.

[Is this why physicists want a path integral?]

That's for quantum mechanics.

So these equations, call these x paths, they satisfy what are called Euler-Lagrange equations. We'll eventually see that these are just Newton's second law. Let me just continue with the philosophical baloney. This sort of variational principle is also found in geometry, where it used to obtain nice PDEs, like the harmonic PDE.

The Lagrangian approach gives us back our phase space, but it gives us a lot more than that. The symplectic form was borrowed and depended on a time t. In the Lagrangian approach, we'll get, the information embedded in this Lagrangian mechanics, which are the Euler Lagrange equations and the submanifold $\mathscr{M}$, but also a family of one-forms on

$\mathscr{M}$ parameterized by time. Finally, these one-forms will give us the symplectic structure naturally, and that won't depend on t.

I've kept you guys ten minutes long, I apologize. But in this sense, physicists equate "theory" with a particular Lagrangian, which has all of this information in it.

[I thought it was the action?]

That's the integral of the Lagrangian, which I think is more basic.