Mathematical Physics
September 11, 2006

Gabriel C. Drummond-Cole

September 11, 2006

[Is the exact sequence from last time secretly Noether's theorem?]

We'll see Noether's theorem later. Let me recap what we've seen so far. So far we've seen particle motion, and the structure of the phase space $\mathscr{M}$ which are paths from the affine time to the target satisfying Newton's second law

$$\{x : M^1 \to X | \ddot{x} = -\mathscr{V}'(x(t))\}$$

This has a right action by $\mathrm{Euc}(M^1)$, a left action by $\mathrm{Isom}(X, \mathscr{V})$ and a for each $t_0 \in M^1$ a natural diffeomorphism $\mathscr{M} \overset{t_0}{\cong} TX$.

We saw something about symplectic geometry on a smooth manifold M^{2n}. This means there is a two form $\omega \in \Omega^2(M)$ such that ω^n is nowhere vanishing (nondegeneracy) and $d\omega = 0$ (closed).

The thing that will play a big role today is the symplectic gradient which takes smooth functions on a symplectic manifold into vector fields $C^\infty(M) \to \mathscr{X}(M)$ via $f \mapsto \xi_f$ characterised by $\iota(\xi_f) = df$. This gives us the Poisson bracket $\{\cdot, \cdot\}$ which makes $C^\infty(M)$ a Lie algebra. This is given by $\{f, g\} = \omega(\xi_f, \xi_g)$. Then this map ξ is a homomorphism of Lie algebras.

The prime example of a symplectic manifold is when $M = T^*X$, the cotangent bundle. Then $\omega = d\theta$ where θ is the God-given one-form on T^*X.

Why is this interesting to us in the context of particle motion? These diffeomorphisms give us a relationship, but we want to get from the tangent to the cotangent bundles. So we use the Riemannian metric to get $\mathscr{M} \overset{t_0}{\cong} TX \cong T^*X$. So now the rest of this class will be spent investigating, take the natural structure on T^*X and pull it back by $\mathscr{M}$. So we have a symplectic structure for each t_0 and these could depend on the choice of t_0. So this is breaking the symmetry.

This brings us to Hamiltonian mechanics. The goal of Hamiltonian mechanics is to encode the symmetries of our phase space into the Lie algebra of smooth functions with the Poisson bracket $(C^\infty(M), \{\cdot, \cdot\})$.

To point out, to be grandiose, where this fits in the grand scheme of physical systems, there's usually a phase space, and another space (of observables). There should be a duality $\sim$ between them, as observables are evaluated on states. In our particular situation in classical mechanics, our state space is our phase space. Our observables are the functions on our phase space. These would be things like momentum and energy that we can assign to a particular particle path.

One would expect that the symmetries of the phase space should translate into symmetries of the symplectic structure. Let me talk about that, and symplectomorphisms. I'm never going to write out symplectomorphism again. I probably spelled it wrong in the first place. I'll call them whatever in the future, unless you want me to call them, like Bob. That might look bad in Gabe's notes.

Weinstein coined the term symplectic, from taking the Greek equivalent for the Latin word for complex. Before it was the Abelian linear group. It sounds like a Victorian word, like perambulator. It's the Greek root for intertwined.

Let (M, ω) be a symplectic manifold. Then $\varphi \in \mathrm{Diff}(M)$ is a symplectomorphism if and only if $\varphi^*\omega = \omega$. Let me give you some examples related to the cotangent space. Since this happened automatically, we might think that any diffeomorphism from a diffeomorphism of the underlying manifold would be a symplectomorphism. That is the case.

If $M = T^*X,\ \omega = d\theta$ and $\varphi \in \mathrm{Diff}(X)$, then

$$\varphi : T^*X \xrightarrow{\sim} T^*X$$

by $(x, p) \mapsto (\varphi(x), (d_x\varphi^{-1}) * p)$. So to check that this is a symplectomorphism, you just check that this preserves θ.

Let's look at a subclass where $X = \mathbb{E}^d$, so $M = V^* \times \mathbb{E}^d$ and let $\varphi = A \in \mathrm{Euc}(\mathbb{E}^d)$. So for $x \in \mathbb{E}^d$ and $p \in V^*$ then $\Phi(x, p) = (Ax, (dA^{-1})^*p)$ is a symplectomorphism.

In the linear category last time this is analogous to the subgroup, we said $GL(L) \subset Sp(L \oplus L^*)$, and this is the general analogue of this linear statement.

The reason I harped on these examples is because when we talked about particles, there are transformations on the target space. The diffeomorphisms will give us special symplectomorphisms on the phase space.

Now I want to talk about infinitessimal symplectomorphisms. So $\xi \in \mathscr{X}(M)$ is an infinitessimal symplectomorphism if and only if $\mathrm{Lie}(\xi)\omega = 0$. This leads us to a special subset of vector fields $\mathscr{X}_\omega = \{\xi \in \mathscr{X}(M) | \mathrm{Lie}(\xi)\omega = 0\}$. This sits inside $\mathscr{X}(M)$ as a subalgebra, preserving the Lie bracket.

So as long as you stick with diffeomorphisms isotopic to the identity, these are the same requirements.

Okay, now the symplectic gradient. For any $f \in C^\infty(M)$ I get a vector field ξ_f. I claim that this lives in $\mathscr{X}_\omega(M)$. To see this note that

$$(Lie(\xi_f)\omega) = d \circ \iota(\xi_f)\omega + \iota(\xi_f)d\omega = 0,$$

because $\iota(\xi_f)\omega = df$ and $d\omega = 0$.

So what if I want to look at a particular symmetry. Can I find corresponding obervables? Does every infinitessimal symmetry have a corresponding observable? The answer will depend on H^1. The short answer is no. The long answer brings up the exact sequence

$$f \longrightarrow \xi_f$$

$$0 \longrightarrow H^0_{dR} \longrightarrow \Omega^0(M) \longrightarrow \mathscr{X}_\omega \longrightarrow H^1_{dR} \longrightarrow 0$$

$$\xi \longrightarrow [\iota(\xi)\omega]$$

So if $[\iota(\xi)\omega] \neq 0$ then ξ has no corresponding observables. If $\xi \in \mathscr{X}_\omega$ has an observable, it has many, only unique up to the constants.

Okay, now time translation. In classical mechanics, there is always a distinguished one-parametery group of time translations. Let's just assume for now that ξ_t is the corresponding infinitessimal generator of time translation, that is, $\iota(\xi)\omega$ is exact.

So we have a choice of corresponding observables. Pick one, up to a constant. Finally we meet the energy. This is the Hamiltonian, which in the sense of these infinitessimal symmetries, is negative the corresponding observable for time translation. In other words, energy, once you take the symplectic gradient, it generates motion which is the negative of time translation. So that is $\{x \mapsto x \circ T_s, s \in \mathbb{R}\}$.

So