Math Physics
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So I know we've been, we're getting into electromagnetism, and we've been throwing the Hodge star operator around, do you guys want to review it?

Now we'll have an algebraic reprieve. This is so that everybody is comfortable with it. The setup is, let V be a real vector space with a nondegenerate symmetric bilinear form $\langle,\rangle$. For now we'll say the dimension is n. I didn't specify that it was definite or anything like that, I want to allow any signature. Okay, so

- $\langle\,,\,\rangle$ defines a pairing on the dual V^*, and in fact, on the entire tensor algebra.

- In particular, it defines a pairing on the exterior algebra. That's what we're going to focus on.

In particular, $\langle\alpha_1 \wedge \ldots \wedge \alpha_q, \beta_1 \wedge \ldots \wedge \beta_q\rangle$, where each of these is an element of $\bigwedge^q V^*$, then we take this inner product to be $\det([\langle\alpha_i, \beta_j\rangle_{V^*}])$ where $[A_{ij}]$ is the $q \times q$ matrik with ij component A_{ij}.

For the sake of computation, it's useful to note that if we start with an orthonormal basis $\{e^i\}$ of the dual, meaning that $\langle e^i, e^j\rangle_{V^*} = \pm\delta^{ij}$.

Now this orthonormal basis generates a natural orthonormal basis for the exterior product $\{e^{i_i} \wedge e^{i_q}\}$.

- The determinant line of V is the top exterior power of V, and I'll write it as $\det V$, meaning $\bigwedge^{top} V$. In particular $\det V^*$ are just the volume forms on V.

- If we have a symmetric nondegenerate bilinear form on V then there are two volume forms ω such that $\langle\omega, \omega\rangle = \pm 1$. They differ by a sign. To talk about the Hodge star, we have to choose one, which is equivalent to choosing an orientation, or a component of the det $V\backslash\{0\}$. The context in which we are discussing the Hodge star operator, we have a symmetric nondegenerate bilinear form and an orientation with its distinguished volume form.

- The Hodge $*$ operator is a linear map $* : \wedge^q V^* \to \wedge^{n-q} V^*$, implicitly defined by

$$\alpha \wedge *\beta = \langle *\langle \alpha, \beta \rangle \omega,$$

where α is a q-form.

Exercise 1 *Check that the Hodge $*$ is well defined.*

Exercise 2 *Compute $**$ The answer is ± 1*

Let's say that V has dimension three and $\langle \ , \ \rangle$ is positive definite. Then take e^i to be a basis of V^*, and $e_1 \wedge e_2 \wedge e_3$ the top form.

Then $*e^1 = e^2 \wedge e^3$, $*e^2 = e^3 \wedge e^1$, and $*e^3 = e^1 \wedge e^2$. This is fairly easy to compute. Now let's take the dimension to be four and the inner product to be Lorentzian, East coast, $+ - - -$.

[How can you remember that?]

[The $+$ looks like land and the $- - -$ look like water.]

Okay, so we have $\langle e^0, e^0 \rangle = 1$, $\langle e^j, e^j \rangle = -1$. Let $\omega = e^0 \wedge e^1 \wedge e^2 \wedge e^3$ Then $*(e^0 \wedge e^1) = -e^2 \wedge e^3$, and $*(e^1 \wedge e^2) = e^0 \wedge e^3$.

Any other questions?

All right, now electromagnetism. So William's presentation so far is that we have a one-form $A \in \Omega^1(M^4)$. Initially this was introduced as something that couples to relativistic paths. This coupling was the pullback $x^* A$. We colud write the Lagrangian and that term appeared in it. The physically relevant guy was $F = dA \in \Omega^2(M^4)$. Now A is allowed to dave singularities, but F is not. The gauge field A is not the physical thing, F is the physical thing.

So for now that's the physically relevant field. In classical electromagnetism, this is all global. The reason this was the physically relevant field was that we had this equation of motion $\iota_{\frac{dx}{dt}} F = (m \frac{d^2 x}{dt^2}, \bullet)$. William pointed out that we can study this guy independently and it's got its own equations of motion and we can write them down, $dF = 0$ and $*d * F = J$.

This is manifestly Poincaré invariant on M^4. This is obvious on the first equation, and $*$ commutes with isometries, so everything still goes through. In the next few weeks we'll throw all that out and deconstruct everything. We'll do undergraduate electromagnetism. We'll do straightforward undergraduate electromagnetism.

So foliate M^n with spacelike $n - 1$ dimensional planes, so that $M^n = \mathbb{A}_{time} \times \mathbb{E}^{n-1}$. The tangent spaces are canonically $V = \mathbb{R} \oplus W$. With respect to this choice we can decompose the two-form, well, we can choose coordinates on $\mathbb{A}_{time} \times \mathbb{E}^{n-1}, dt, dx^1, \ldots dx^{n-1}$. Then F decomposes into things that involve dt and things that don't. So we can write this as $dt \wedge E + B$. So $E(t) \in \Omega^1(\mathbb{E}^{n-1})$, and notice that it is time dependent. B is in $\Omega^2(\mathbb{E}^{n-1})$. These are maps from affine time into forms.

Exercise 3 *Prove Maxwell's equations for $F = dt \wedge E + B$, namely*

$$dB = 0, \qquad dE = -\frac{\partial B}{\partial t}, \qquad d*_{\mathbb{E}^3} = \rho, \qquad c^2 d *_{\mathbb{E}^3} B = *_{\mathbb{E}^3} \frac{\partial E}{\partial t} + j.$$

Some cases are, well, for a constant electric field we would have $B = 0$ and $\frac{\partial E}{\partial t} = 0$. Then $d *_{\mathbb{E}^3} E = \rho$, $dE = 0$.

Homework:

$\langle b, **b \rangle \omega = b \wedge ***b = (-1)^q ***b \wedge ****b = (-1)^q \langle ***b, ***b \rangle \omega = (-1)^q \langle *b, *b \rangle \omega = (-1)^q *b \wedge **b = (-1)^q *b \wedge b = b \wedge$

$$(**b, b)\omega = b \wedge *b = (-1)^{q(n-q)} * b \wedge ****b = (-1)^{q(n-q)} (*b, ***b)\omega$$
$$= (-1)^{q(n-q)} ***b \wedge **b = **b \wedge ***b = (**b, **b)\omega$$