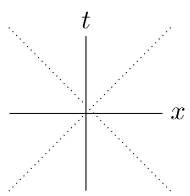
Mathematical Physics
October 3, 2006

Gabriel C. Drummond-Cole

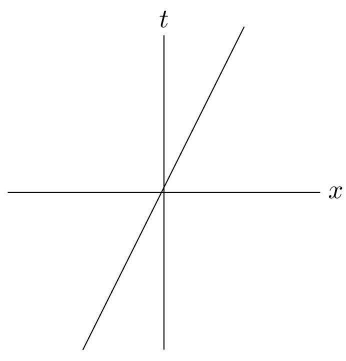
October 3, 2006

Let's try to pick up where we left off. We're using space time diagrams to construct in a geometric way the Lorentz transformations and deduce physical consequences without any algebra. After we get past this we'll see everything in terms of linear algebra and then things will go faster. So if you have questions about the physical aspects you should ask them because after this it will be pretty abstract.

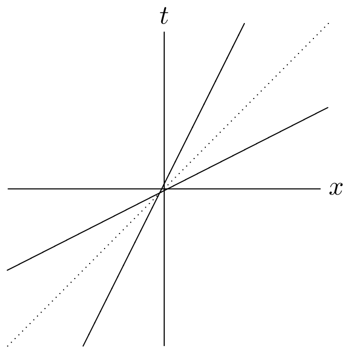
So we're talking about Spacetime diagrams, which should be thought of in (3+1) dimensions, so there are horizontal spatial axes and a vertical time axis. The speed of light is one.



So we want to look at another observer.



We found out that according to this other second observer, the time=0 was the reflection of this path through the $c = 1$ axis.



So we found the $t = 0$ axis of $\mathscr{O}'$. At time $-t$, the observer sends out two light pulses to equidistant mirrors; at time 0 the light hits the mirrors, and at time t the observer sees the light pulses return. So the event A at the location of the observer and B of the light hitting the mirror occur at the same time according to $\mathscr{O}'$, but not according to $\mathscr{O}$. In fact, according to $\mathscr{O}$, $t_B > t_A$ strictly.

Any questions so far?

Now we're going to recalibrate. We'll define something called the invariant interval. We'll start by defining an interval. This is $ds^2 \equiv -dt^2 + d\vec{x} \cdot d\vec{x}$. The important thing is if $ds^2 = 0$ in $\mathscr{O}$, then $ds^2 = 0$ in all $\mathscr{O}'$. That's essentially because, if you plug it in, you get $\frac{d\vec{x}}{dt}^2 = 1$.

[Is x a particle or an axis?]

It doesn't have to be, here it's just coordinates in spacetime. Use the origin as a reference if you have problems anywhere else.

[Is light the only thing that moves at that speed?]

No. The claim is that light all travels at the same speed and we all agree on that speed. It's the same statement for any particle that has no mass.

Those of you who know about tensors, this is a case of the statement that a tensor which is zero in one coordinate system is zero in all coordinate systems.

Now we come to the **Fundamental Theorem of Special Relativity**: If $ds^2 = \ell^2$ is a constant in $\mathscr{O}$, then $ds'^2 = \ell^2$ in any $\mathscr{O}'$.

I'm actually defining a tensor $ds^2 = \eta_{ab} dx^a \otimes dx^b$, where $\eta_{ab} = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \delta_{ij} \end{array}\right)$. In general relativity, you replace this flat constant metric with a more general bilinear form, $g_{ab}(x) dx^a \otimes dx^b$. So the statement is just that this is a tensor. So Lorentz transformations are precisely $SO(3,1)$, precisely the things that preserve this form.

Let's prove this geometrically.

1. $ds'^2 = \underbrace{\phi(\vec{v})}_{\text{conformal factor}} ds^2$. Since these things differ by a constant velocity, it should be clear that the new coordinates have to be related to the old ones linearly (affinely), meaning $(x')^a = \Lambda^a_b x^b + \epsilon^a$. This could be written $\vec{x}' = \Lambda\vec{x} + \vec{\epsilon}$ but that suggests three dimensions. This is called a Poincaré transformation, or an imhomogeneous Lorentz transformation. Poincaré saw that Maxwell's equations [unintelligible]symmetry group of any Euclidean [unintelligible]. Einstein thought that maybe Newton's laws were an approximation with $c = \infty$. Lorentz wrote down the transformations that preserved Maxwell's equations.

   The ϵ is trivial because everything is invariant under shifting so I can forget it.

   So now I need, because of this, the most general expansion is $ds'^2 = M_{ab}dx^a dx^b = M_{00}dt^2 + 2M_{0i}dtdx^i + M_{ij}dx^i dx^j$. Go back to the special case where $ds'^2 = 0$. From linear algebra, if this is true for any direction x^i, this implies that $M_{0i} = 0$ nad $M_{ij} = -\delta_{ij}M_{00}$. Then $ds'^2 = -M_{00}(-dt^2 + d\vec{x}\cdot d\vec{x}) = \phi ds^2$ where $\phi = -M_{00}$.

2. $\phi(\vec{v}) = 1$. How do we get this? Well, it's rotationally invariant so $\phi(\vec{v}) = \phi(|\vec{v}|)$.

   If you rotate so that $\mathscr{O}$ and $\mathscr{O}'$ are in the same plane, then you can put events in perpendicular to the plane spanned by them, which are simultaneous in both frames. The point is that if the boost, the boost doesn't affect any of the lengths orthogonal to the boosts. The length of the rod perpendicular to their plane, according to $\mathscr{O}'$, squared, is $\phi(\vec{v})$ times the length squared according to $\mathscr{O}$. It will change according to the size of the velocity, but it shouldn't depend on the direction.

   Now we want to define an $\mathscr{O}''$ frame. So $\mathscr{O}'$ has the property that it is moving at velocity v relative to $\mathscr{O}$. Define $\mathscr{O}''$ as moving at velocity $-v$ with respect to $\mathscr{O}$. This might look funny. S now $ds''^2 = \phi(-\vec{v})ds'^2 = \phi(-vecv)\phi(\vec{v})ds^2$. So these are both equal to $\phi(|\vec{v}|)$ so this is $\phi^2 ds^2 = ds^2$, meaning $\phi^2 = 1$. Now ϕ has to be plus one so that the length is a positive number.

   So $ds'^2 = ds^2$, so it's called the invariant interval. We'll see that this thing is not general, I didn't say the form of Λ^a_b.

Now what? Now we can consider invariant hyperbolae. Anything where ds^2 is a constant is preserved here. In particular, $t^2 - x^2 = 1$ is preserved. This is a hyperbola. also $t^2 - x^2 = -1$, and so on. In particular, you have a picture, well, you have hyperboloids for ds^2 constant. These are called mass shells. Only the two-sheeted ones are mass shells. In general they're all called mass shells. The invariant distance restricted to them is constant. You can rotate any point on a one-sheeted hyperboloid to any other. So here $ds^2 = 0$ and this is called spacelike seperation. If you're in the two-sheeted, you have $ds^2 < 0$ and you are timelike. So there's a concept of forward and backward. One is known as the causal future of the origin. Any light ray emitted from the origin goes forward on the cone, and is called lightlike.

So, um, back to the question of why $SO(3,1)$ instead of $O(3.1)$. If I chose $O(3,1)$ I could get some things that take my future to my past. Discrete Lorentz transformation are not connected to Lorentz transformations. Also, we have the forward and backward light cones. The causal future is also called the interior of the forward light cone.

This is all words, but we'll probably get to use them. It's called causal because no signals can propagate from one place to another faster than the speed of light.

Now I want to show the graphical construction for the calibration of the axes. Showing it in pictures is pretty easy.

Now we can say how time is ticking on his clock. The invariant interval ticks in the same way on any one of his shells. I have my one second according to me. So in this case, I'm sorry, I should say one more thing. There's an alternative concept $d\tau^2 = dt^2 - dx^2 = -ds^2$. If you go to $\mathscr{O}$'s rest frame, then at the origin $x = 0$ and $dx = 0$ so $d\tau^2 = dt^2$. If $d\tau^2$ is one second, then this is invariant and behaving like a time. This is called proper time. It's an invariant property under Lorentz transformation. So one second is on this hyperbola for him.

[A question about boosts.]

A rotation takes space to space. Boosts mix space and time.

What are the effects of this? The two observers disagree on what they call time and what they call space. Okay, so, time dilation. So c seconds, according to $\mathscr{O}$, took $\mathscr{O}$ just sitting still to another time. Then there's a hyperboloid passing through B. Put A at the origin. Then $c = -t^2 + x^2$. Then what is $(\Delta t)_{AB}$? We want $(\Delta t')_{AB} = c$. So at B, we have $c^2 = t_B^2 - x_B^2 = t_B^2(1 - \underbrace{\frac{x_B}{t_B}}_{v})^2$.

What I have then is $t_B = \frac{1}{\sqrt{1-v^2}}c$. This factor is γ. Since $v \in [0, 1)$, we have $\gamma \in [1, \infty)$. The time intervals that the $\mathscr{O}$ frame is seeing are always larger. If someone is approaching the speed of light, their time is not moving at all.

$\mathscr{O}'$ is doing something simple, just measuring when an event happens. You're doing something complicated, comparing two times on two different clocks. Nevertheless, if your sitting on earth and your twin goes off to space, you'll see him aging slowly.

For Lorentz contraction, I get the x' axis and consider a rod that's at rest with respect to $\mathscr{O}'$. As this rod passes $\mathscr{O}$, what does it look like? It's going to lie in $t = 0$, so it's shorter. So then $\frac{t_C}{x_C - x_B} = \frac{1}{v}$. Similarly, $\frac{t_C}{x_C} = v$. We found that $x_C = \gamma\ell$. So you put it all together and you get that according to $\mathscr{O}$, the length x_B is $\frac{\ell}{\gamma}$. The length according to $\mathscr{O}$ is shorter by a factor of γ.

If that wasn't too fast, next time we'll go to algebra. This is the basic stuff that everyone talks about. Maybe we'll give you some homework.