Math Physics
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Gabriel C. Drummond-Cole

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Last time the idea was that we can model, irreducible representations of the Poincaré group, $ISO(3, 1)$, were labelled with nonzero mass and spin s or zero mass and helicity. We won't talk about this until quantum mechanics but later on we talk about states, instead of functions on a cotangent bundle being vectors on a Hilbert space. So then you'll see more directly the connection between s and what you might think spin means, basically the idea of a ball spinning on its axis so that this is the analogue of angular momentum. This has a discrete number of states and this keeps track of how many. But, so, we were getting to classical field theory, either by loooking at particles working in background fields, or by looking at representations of the Lorentz group and saying that a classical field is a map from spacetime into some target, be it $\mathbb{R}$, cC, or whatever, a section of a line bundle, and so on. The one we're going to takes values in, there's the associated $U(1)$ bundle where the Maxwell field sits, and so on. So this one looked locally like a one-form. There can be gerbes that looked locally like a two-form.

You can take a $Spin(3, 1)$ bundle over $\mathbb{R}^{3,1}$-bundle and you get Ψ_a a Dirac Spinor. Here you would have a spinor index, this transforms in a certain way, then the one-form we're talking about transforms in a certain way, these structures are things that you tack on to this. There are sections over Minkowski space, and some subspace corresponds to irreducible representations.

If you have functions on the real line those give you translations on the real line by differentiation. In the same way we have sections over Minkowski space being representations of the Poincaré group. This acts on this by right composition. That gives us a big representation. Then you look at the direct components.

So okay, maybe we can have another discussion about that, the more correct mathematical statement, but for now let's move on to electromagnetism. What we did previously, we looked at the point particle action in curved background and a one-form background. Let me do the massive case. I don't want to deal with the einbeins. The action was

$$S_{\text{point particle}} = \int d\lambda\{\frac{1}{2}g_{ab}(x)\dot{x}_a\dot{x}_b + qA_a(x)\dot{x}^a\}$$

Since I don't want a curved background I'll simplify my life and take a flat background, Minkowski background, $g_{ab} = m\eta_{ab}$.

Then we derived the momentum conjugate to x^a, which was $p_a = m\dot{x}_a + qA_a(x)$. This implies, of course, that $\dot{p}_a = m\ddot{x}_a + q\delta_b A_a \dot{x}^b$. The equations of motion, which is, this thing differentiated with respect to x and then take the time derivative. So you say $\dot{p}_a = \frac{\partial L}{\partial x^a} = q\partial_a A_b \dot{x}^b$. So we get

$$m\ddot{x}_a = qF_{ab}\dot{x}^b$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$. I started with, this was designed to give the correct nonrelativistic limit.

Now I can point out that were I to change the one-form to a different one-form related to the first by a total derivative, $A_a(x) \mapsto A'_a(x) = A_a(x) + \partial_a \alpha(x)$.

If I were to put it in your language the statement would be obvious, I'm doing $\int x^*(A)$, so if you change A by $d\alpha$ that integral doesn't change. So $F = dA$. This means A is defined everywhere, which I don't always want to assum. Another way of saying this thing about changing the forms is that this change doesn't change the field strength, so it's a gauge symmetry. If I wrote $A' = U^{-1}AU + U^{-1} + U^{-1}dU$, in the Abelian limit, you would get this change.

Now this is just a recap of what we talked about before, these are point particles in a fixed background. What about dynamics for A_a? So we can try to just use intuition from the point particle. There we needed second order in derivative equations of motion to give it the right observable properties. Also if you have too many derivatives you will have problems with forward time propagation of Cauchy data. It's at lowest order linear in the coordinate of the particle. We defined an action in the classical case, can we do something similar here?

We have translation invariance, covariance with respect to Lorentz transformations, and gauge invariance. This says that at least one of the components is zero, by changing coordinates. You can set $A_0 = 0$ but you break Lorentz invariance. You could instead constrain the divergence $\partial \cdot A = 0$. You could remove one degree of freedom when you describe the vector field. If the dynamics of A don't describe this in the same way, I would have an extra degree of freedom. A physicist thinks that the dynamics A_a should keep gauge invariance.

What about

$$L = \partial_a A_x \partial^a A^b?$$

This is quadratic in A and second order in derivatives but it doesn't satisfy gauge invariance.

The obvious thing to do is write this in terms of F. This is linear in ∂ and A and gauge invariant. So what about $L = -\frac{1}{4}F_{ab}F^{ab}$. That should work. In fact, it will turn out to be the only thing that will work. So this is $-\frac{1}{4}F \wedge *F$.

Okay, so we define

$$S = \int d\lambda \{\frac{m}{2}\dot{x}^2 + qA_a(x)\dot{x}^a\} - \frac{1}{4}\int d^4 g F_{ab}(A(g))F^{ab}(A(g))$$

Okay, so now I need to do something a little bit painful. Let me make my life easier. Since we already know all the equations of motion, let me just do the parts for A. So let me qust do the piece that looks like

$$q\int d\lambda A_a(x)\dot{x}^a - \frac{1}{4}\int d^4 g F^2,$$

where F^2 means with respect to the inner product.

Now I need to rewrite this over all of spacetime. So

$$\begin{aligned}\int d\lambda A_a(x)\frac{dx^a}{d\lambda} &= \int d\lambda A_a(x)\frac{dx^a}{d\lambda}\int d^3 g\delta^{(3)}y^b - x^b(\lambda)) \\ &= \int d\lambda d^3 y A_a(y)\underbrace{\left(q\frac{dy^a}{d\lambda}\delta^{(3)}(y-x(\lambda))\right)}_{j^a(y)}.\end{aligned}$$

Great, now I want to parameterize so that I have the right coordinates, and I can write it as

$$\int d^4 y A_a(y) j^a(y).$$

That's what I'm doing. So the action is

$$S = \int d^4(x)\left\{A_a(x)j^a(x) - \frac{1}{4}F_{ab}F^{ab}\right\}.$$

So J is a distribution of current, a current density that doesn't prima facie depend on a particular point in space. That's the sense in which this thing is localizing on the physical world line. It's a current because it depends on the derivative, it's moving charge. q is charge and the velocity is the derivative next to it.

Let's move on. The equations of motion are

$$\frac{\delta S}{\delta A^a(x)} = 0.$$

Here this gives

$$j_a(x) - \frac{1}{2}\int d^4 y\left(\partial_c\frac{\delta A_d(y)}{\delta A^a(x)} - = partial_d\frac{\delta A_c(y)}{\delta A^a(x)}\right)F^{cd}$$

Then I can combine the two terms because F^{cd} is antisymmetric, so I get

$$j_a(x) - \int d^4 y\delta_c\left(\delta^{\{}(4)\}(x-y)\right)F^c_a.$$

I assume that any field vanishes sufficiently rapidly at infinity.

Then I can integrate by parts and I get $j_a + \partial^c F_{ca}$. The equation of motion is then $\partial^b F_{ab} = j_a$. Okay?

Next time we'll write this in a more familiar way, to Maxwell's equations. This is two of them. Remember the assignment efore was to take $m\ddot{x}_a = qF_{ab}\dot{x}^b$ and define $E^i = -F^{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ So you can do $\vec{F}(\vec{E} + \vec{v} \times \vec{B})$. Plug in E and B to work this out. You get the other two from $\partial_{[a}F_{bc]} = 0$.