Math Physics
October 24, 2006

Gabriel C. Drummond-Cole

October 24, 2006

So, uh, are there any questions? There should be a lot of questions. Last week Jerry tried to explain what I was explaining in terms of, we talked about the Lorentz group and then talked about the commutation relations that defined the Lorentz algebra. I was trying to do this from the point of view of classical mechanics extended. I wroted down this explicit formula

$$MSab = x_a p_b - x_b p_a.$$

We worked out the Poisson bracket and found out that

$$\{M_{ab}, M_{cd}\} = \eta_{c[a} M_{b]d} - \eta_{d[a} M_{b]c} = g^{ef}_{ab,cd} M_{ef},$$

which I described as the structure constants of $SO(3,1)$. Here the η are signed Kroenecker deltas; if they had been actual δs then this would have been $SO(4)$. Really what I was after was a general statement about representations of $SO(3,1)$, which are maps $g : G \to GL(V)$. So then I went back to look and see what these things look like on Poisson brackets with f. So $\{M_{ab}, f(x)\}_{\text{P.B.}} \mapsto [\hat{M}_{ab}, f(x)]$, where $\hat{M}_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \partial\partial x^a$. This will give the exact same structure constants. So here I'm actually picking V to be smooth functions on the manifold. So if we pick $V = C^\infty(M)$, then this is what I was calling a field representation. This is $SO(3,1)$ because I have M_{ef}.

What else can we take, we can use $\mathbb{R}^{3,1}$ for our manifold, but what else can we do? We could take $V = \mathbb{R}^{3,1}$. So here we had $\hat{M}_{ab} = x_a \delta_b - x_b \delta_a$. Since this acts on vectors, this needs indices which are vectorlike. So we have $(\hat{M}_{ab})^c_d$. Then $v^a \mapsto \frac{1}{2}\omega^{cd}(M_{cd})^a_b v^b$. As a homework assignment I said to check that if you put in $i(\delta^c_a \eta_{bd} - \delta^c_b \eta_{ad})$, you get the same structure constants. So the transformations, raising and lowering indices, is what you'd expect, $w_a \mapsto (\frac{1}{2}\omega \cdot M)^b_a w_b$. So you can extend this to tensor products in a natural way, taking $V = M^{\otimes n}$.

Now I want to give a homework assignment. This seems to always be a confusion. Take a complex vector space $V = \mathbb{C}^4$, and define γ^a acting on V for $a \in \{0,1,2,3\}$. This being four dimensions, range over $\{1,2,3,4\}$. We want to satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab} I$. Physicists call this a Clifford algebra. It's not the same as what mathematicians call a Clifford algebra. So now define $(\hat{M}_{ab})^\beta_\alpha = i[\gamma_a, \gamma_b]^\beta_\alpha$. The claim is that this is a representation of $SO(3,1)$ in the sense that the vectors for a representation of $SO(3,1)$, and the M maps obey the relations of the Lorentz algebra. This representation is called the spinor representation. The thing that this acts on is something Ψ_α, which is called a spinor.

If you're interested in this, take $\gamma^0 = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ where these are two by two blocks, and then

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

check that these satisfy the desired relations for the γs. So there are representations of $ISO(3,1)$. Here is a fact. The first fact is if you have a combination of generators that commute, let me say what the generators are. These are the momenta and the M_{ab}. You can think of taking these things and monomials in them. How many of these things, I want to find things that commute with p_a and M_{ab}. I want the smallest number of operators that commute with the generators. p^2, for instance, commutes with p, and also commutes with M_{ab} because this is a scalar. You can check this explicitly at home. So p^2 commutes with both of these. Obviously there are infinitely many of these, but there are $rk\ G$ such commuting operators.

So for $ISO(3,1)$, this should be rank two. One is p^2, which is the Casimir element, which is T^iT_i where T^i generate the Lie algebra. So this is one of them. Physicists call this the "quadratic" Casimir operator. I think that you can probably write all the Casimirs as quadratic ones. As physicists we don't want to do this because it might break Lorentz invariance.

Here is a quartic Casimir operator. First let me define the Pauli-Lubanski (pseudo-)vector $w_a = -\frac{1}{2}\epsilon_{abcd}M^{bc}p^d$. So first, $[\omega_a, p_b] = 0$, because the momenta commute with each other, and rotations take momenta to one another. What is obviously not the case is $[\omega_a, M_{bc}] \neq 0$. We know that one-forms are not invariant under rotations and boosts. So we need to define w^2, which is order four in the generators. We squared w to make it commute with M. I can't make a transformation to make w^2 look like p^2. We have two commuting operators ("Casimirs"). So p^2 and w^2 are obviously linearly independent, and there are two of them. The claim that I didn't prove is that I have a complete set when I had two of these things. So what's the point? These commute with every generator of the Lie algebra. Then Schur's lemma says that on any irreducible representation these things have to be proportional to the identity. So p^2 is proportional to the identity, and we'll just say that p^2 has as its constant m^2. There might be a minus sign. So $p^2 = \lambda_p I$, and we know that $p^2 = -m^2$ was the mass shell condition. So where I'm going with this, we can use this Casimir set to label a representation. Now, whether they're the masses or the angular momenta or whatever, I don't know. So p^2 is $-m^2$. We also went to the rest frame and found that p^0 was e. If we go to the rest frame, then positive energy is $p^a = (E,0,0,0)$ So we could switch the energy to negative. To go into your representation, you need the Hamiltonian to be bounded below, you need positive energy, a vacuum state. So a positive energy requirement gives you this mass sign. By physical requirements, suffice it to say, the number λ_p is $-m^2$. What about

$w^2 = \lambda_w I$. This is $\pm m^2 \vec{J}J$, where $\vec{J}^i = \frac{1}{2}\epsilon^{ijk}M_{jk}$. If you have λ_p you have the rest frame, you can always do a boost so that p^a is $(m, 0, 0, 0)$, and then just plug it in. You get a factor of m when the other factors are spatial. So you coan see how w_a is basically mJ. Then this eigenvalue is called spin. Actually it's $s(s+1)$, and the s is called the spin. So you can plug in explicitly the vector representation, and you find that $s(s+1)$ is 2, so that photons are spin one. When the mass is zero these things are not linearly independent, but if these were massive this would be spin one.

With $m = 0$ the problem is that $p^2 = 0 = w^2$ so they're not linearly independent. But it's clear from what we said before that $pw = 0$. Then by some other brilliant guy's lemma, that means p is proportional to w, so that $w_a = \lambda p_a$. Then λ is called the helicity, which is the same thing as spin, but for massless particles. So then you go through your thing again and find out that this is like spin.

So take the momentum, it moves on a line. You're projecting onto your direction of motion. It's the projection of the angular momentum along the direction of motion. Normally you could say that's meaningless because you could change the direction. Since it's light you can't do that, so once you've computed this it stays that way. This is a property of the photon, not the reference frame.

Now I was going to start talking about fields. We went through this thing where, we came to this notion of fields like A_a, the one form, and the metric, g^{ab}. These didn't have any dynamics. We worked out Lorentz's law. So then we stopped and went back to the Lorentz group, went to its algebra and its representation, I didn't mention it explicitly, these should be thought of as functions on M that transform like these representations. So these would be vector fields. The natural place to go now would be fields in general and their dynamics. Now we want to promote to the kinematics of these fields themselves, so that we don't fix them as backgrounds, but study their properties. We'll work on electromagnetic fields. We found that the particle obeyed a force law $F_a(x) = qF_{ab}(A(x))\dot{x}^b$. We took a massive free particle with action

$$(\frac{m}{2}\dot{x}^2 + qA_a(x)\dot{x}^a)d\lambda.$$

And I'm saying that $m\ddot{x}^a = qF^{ab}\dot{x}^b$, where here $F_{ab}(A(x)) = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} = \delta_a A_b - \delta_b A_a$. So here you got $\vec{F}_{em}q(\vec{E} + \vec{v} \times \vec{B})$, where $E^i = -F_{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$. You can see that E and B obey certain principles themselves. We take a small number of derivatives, Lorentz invariance, and we get relations among these, Maxwell's invariants.