Math Physics
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So this is another one in the series of what William said.

He's been talking about the representation theory of the Poincaré group.

We're working with M^n, which is affine Minkowski space over a Lorentzian inner product space V. Let $x^0, \ldots, x^{n-1}$ be an affine, orthonormal set of coordinates. So g is $dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - \ldots - dx^{n-1} \otimes dx^{n-1}$.

No we have that $\frac{\partial}{\partial x^0}$ is timelike, meaning $||\frac{\partial}{\partial x^0}||^2 > 0$, and the other coordinates $\frac{\partial}{\partial x^i}$ is spacelike, meaning $||\frac{\partial}{\partial x^i}|| < 0$.

So on V you have the light cone C which we use to define certain subgroups. Let $SO(V)$ denote the identity component of $O(V)$ with respect to the Lorentzian metric. So these are $\{A \in O(V) | \det A = 1$ and A preserves the components of $C\}$.

So then there's the Poincaré group P_n, which is the identity component of $Isom(M^n)$, which is in the short exact sequence

$$1 \to V \to P_n \to SO(V) \to 1$$

Physicists prefer to work with representations of the lie algebra. Let $\mathfrak{p}_n = Lie(P_n)$. Then we have the short exact sequence

$$0 \to V \to \mathfrak{p}_n \to \mathfrak{so}(V) \to 0$$

[How is V a Lie algebra?]

It's the Lie algebra of the Lie group V. It's Abelian. It's not like Bill Gates, who's Abelianaire

Anyway, William wants to talk about representations of $\mathfrak{p}_n$. Before that he talked about the structure of this group. He did this in a very physics way. All right. So P_n acts on M^n. It also acts, in a natural way, on T^*M^n. I'm doing this because this is how William was presenting this. He was using Poisson brackets, so I need to do this in the symplectic case.

If $A \in P_n$, then $A(x,p) = (Ax, (dA^{-1})^* p)$. With respect to the coordinates that we're using, we have coordinates on T^*M^n which are $\{x^0, \ldots, x^{n-1}\}$ and then the $\{p_0, \ldots, p_{n-1}\}$ where $p_j : T^*M^n \to \mathbb{R}$ via $p_j(dx^i) = \delta^i_j$, with the form $\omega = dx^i \wedge dp_i$. We have this symplectic structure so we have the Poisson bracket $\{f, g\} = \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}$. We have this explicitly because we've chosen coordinates.

Recall that if G is a Lie group that acts as symplectomorphisms of (M, ω), then there is a map from $(\mathfrak{g}, [\ ,\ ])$ into $(C^\infty(M), \{\ ,\ \})$. That map is the Noether charge, where you take j to the Noether charge of j, which is a smooth function.

William was essentually using this paradigm when he was talking about the Poincaré group. Any questions so far? Well, let's see. the Noether charge for travel along the e_i direction is p_i. The basis element e_i gets sent to the function $p_i \in C^\infty(T^*M^n)$. Here e_i corresponds to x^i.

So William was using the fact that translations are represented by momentum functions. So in particular, if you want to track how translatinons act on $C^\infty(T^*M^n)$, if f is such an object, then $Lie(\frac{\partial}{\partial x^i})f = -\{p_i, f\}$. Okay?

So those are the translations. What about $\mathfrak{so}(V)$. What does this Lie algebra look like, more or less? Say $a \in \mathfrak{so}(V)$. Then I can infinitessimally exponentiate this to $\exp(ta)$. Then $g(\exp(ta)v, \exp t_a w) = g(v, w)$. So differentiating you get $g(av, w) + g(v, aw) = 0$. So this algebra should be naturally isomorphic with $\wedge^2 V$, where $v \wedge w \mapsto g(v) \otimes w - g(w) \otimes v$, where here g is a map $V \to V^*$.

This brings us to $\{e_i\} \rightsquigarrow \{e_i \wedge e_j\}$, a basis for $\wedge^2 V$, which leads to $\{e^i \wedge e_j - e^j \wedge e_i\}$. Now define M_{ij} to be the Noether charge for $e_i \wedge e_j$. Did we compute this? The answer was $x_i p^j - x_j p^i$.

Let ξ_{ij} be the vector field of T^*M^n induced by $e_i \wedge e_j$, then $Lie(\xi_{ij})f = -\{M_{ij}, f\}$. Here

$$\xi_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$$

The Poincaré group acts, I claim, on the tangent space, this is homework. The homomorphism we have from $(\mathfrak{p}_n, [\ ,\ ]) \to (C^\infty(T^*M^n), \{\ ,\ \})$ is injective. Therefore, to look at the structure in $\mathfrak{p}_n$ it suffices to look at its image in $C^\infty(T^*M^n)$. Physicists think of symmetries as functions, associate them with charges. This is a very physics-y thing to do. Should it be physicsie?

[Like Dixie?]

[Dixie is spelled with a y.]

No it's not, you should know, you're from Texas. Texas, with its football, and its..., presidential candidates, and its death penalty for retards.

We've picked a basis. For translations we have e^i and for rotations we have $e^i \wedge e^j$. So let me say that again. A basis of V, which are the e^i, determine a basis for the Lie algebra $\mathfrak{p}_n$, where translations have as basis $\{e_i\}$ and as rotations $\{e_i \wedge e_j\}$. This works because there is a natural splitting of the short exact sequence on the level of Lie algebras (into a semidirect product $\mathfrak{so}(V) \rtimes V$), where $[a, v] = av$.

[What is William's curly δ?]

It's the Lie derivative. Physicists usually are thinking about one particular vector field all the time when they say that.

Okay, so the whole point was that William starts with a basis on V. This gives me a basis on the Lie algebra, and then we have $\mathfrak{g}, [\ ,\ ]$, and a basis $\{T_i\}$. Then $[T_i, T_j] = \Gamma^k_{ij} T_k$, and these are called the structure constants of $\mathfrak{g}$ with respect to the basis $\{T_i\}$.

[How do those transform with change of basis?]

How it transforms depends on where it lives, so it's in $\wedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$, which explains how it transforms with respect to automorphisms of $\mathfrak{g}$.

Because William works with a basis, he expresses everything in terms of structure constants, and does it on the function side.

[What about representations?]

William hasn't talked about those yet much. The map into $C^\infty(T^*M^n)$, that's a representation. If you want it to act on vectors, you choose some matrices in $\mathfrak{gl}(n)$ with the same structure constants. You have V and its tensor algebra, you get those actions of $\mathfrak{so}(V)$ for free. If you want the Lie algebra action to be unitary, you get into quantum field theory.

There's a theorem that for noncompact groups there are might not be a finite dimensional unitary representations. Those functions are fields in physics language.

[Are the functions Lie algebras of an obvious Lie group?]

Hamiltonian diffeomorphisms.

Okay, I was going to say, now we can write $\delta f(x)$. Of course, with respect to some set of constants ω^{ab} which are describing something about the Lorentz transformation. So ω^{01} would be some kind of boost, and ω^{12} would be a rotation around the z axis.

So

$$\delta f(x) = \{\frac{1}{2}\omega^{ab} M_{ab}, f\},$$

where $M_{ab} = x_a p_b - x_b p_a = x_{[a} p_{b]}$. So this whole thing is

$$\frac{1}{2}\omega^{ab}\{x_{[a} p_{b]}, f(x),$$

of which only one term can survive, which is

$$\frac{1}{2}\omega^{ab} x_{[a}(-\frac{\partial p_{b]}}{\partial p_c}\frac{\partial f}{\partial x^c}) = -\frac{1}{2}\omega^{ab}(x_a \delta^c_b \frac{\partial f}{\partial x^c} - x_b \delta^c_a \frac{\partial f}{\partial x^c}$$

$$= \frac{-1}{2}\omega^{ab}\underbrace{(x_a\frac{\partial}{\partial x^b} - x_b\frac{\partial}{\partial x^a})}_{\xi_{ab}} f.$$

So you can replace M_{ab} with ξ_{ab} and $\{\ ,\ \}_{\text{P.B.}}$ with $[\ ,\ ]$.

I think we went over again.

[But only by a minute.]

I should say that if you compute $\{M, M\} = fM$, you'll get the same f with $[\xi, \xi] = f\xi$. That's the point. They're isomorphic as Lie algebras.