Mathematical Physics
October 17, 2006

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Are there any questions? A few, maybe, lectures ago we had talked about Lorentz transformations as rotations and boosts. These form a group. Maybe we did talk about this, it's called the Lorentz group. A rotation and a boost give a boost, and so on. This is called $SO(3,1)$ or $(1,3)$, I can never make up my mind.

This also has a smooth structure, so it's a Lie group. If I had, for example, a general Lorentz transformation, say a rotation of angle θ, if θ is very small, you can expand this in terms of θ, and you get as a first approximation the identity, and then something of order θ, and so on.

Modulo all of the caveats, if you have a Lie group, you can take the tangent space at the identity, $T_{\mathbf{1}}G$, which is called the Lie algebra g You have an antisymmetric pairing $[\ ,\ ],$: $g \times g \to g$. Today we want to look at infinitessimal Lorentz transformations.

Let us recall, for very small velocities, we had this matrix $\Lambda = \begin{pmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v & 0 & 0 & 1 \end{pmatrix}$, which was a boost in the three direction.

So close to the identity, for g in the Lie group, you can write

$$g = \exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \approx 1 + A + O(A^2)$$

Now, we can write the boost as $\mathbf{1} + \omega$, where $\omega_3^0 = -v = \omega_0^3$, and then we write $\omega_{03} = -v = -\omega_{30}$.

Did we do $M_{ab} = x_a p_b - x_b p_a$? We write this as $x_{[a}p_{b]}$ Recall that we had a Poisson bracket $\{x^a, p_b\}_{\text{P.B.}} = \delta^i_j$. Jerry did this for the cotangent bundle of $\mathbb{R}^d$. We're doing this on $\mathbb{R}^4$ with some funny signs. So I'm claiming I can extend this to the new coordinate $\{x^0, p_0\} = \delta^0_0$.

Let's do a sample computation. It's easy to show

$$\{M_{ab}, x_0\} = \{x_{[a}p_{b]}, x^c\} = x_{[a}\{p_{b]}, x^c\} + \underbrace{\{x_{[a}, x^c\}p_{b]}}_{\text{zero}} = -x_{[a}\delta^c_{b]}.$$

You can find also that $\{M_{ab}, p_c\} = \eta_{c[a}p_{b]}$.

Now I want to compute $\{\frac{1}{2}\omega^{ab}M_{ab}, x^c\} = -\frac{1}{2}(2\omega^{ab}x_a\delta^c_b$.

If you remember from classical mechanics, symmetries were generated by conserved charges, and we claimed that the rotational symmetry was generated by angular momentum. You were supposed to take rotational independence in three dimensions, and show that there was a conserved charge given by $\vec{x} \times \vec{p}$. Now all I'm doing, this can be rewritten by taking the i component of this vector and you have

$$\vec{J^i} = \epsilon^{ijk}x_j p_k.$$

Now I can contract on both sides with an ϵ to get $\epsilon_{imn}J^i = \epsilon^{ijk}x_j p_k \epsilon_{imn}$. If you call this J_{mn}, then you get $J_{mn} = x_m p_n - x_n p_m = x_{[m}p_{n]}$. So now we just add time to all of this. So we can use this language, and then what does a rotation look like? If you rotate by θ^i around the i axis, then the infinitessimal change $\delta x^i = \{\theta_j J^j, x^i\}$. So the claim is that $\delta \doteq \{\theta \dot{J}, \dot{\}}$. The translation generator was p, so if you want to translate by ϵ, you do this by $\{\epsilon^a p_a, \dot{\}}$.

Now you can see why I'm inverting the ϵ, because the bracket is easy to write down relativistically, whereas the coordinates aren't.

Now we understand the next step. In classical mechanics you only have two variables, so there's only a few things you can try. Okay. Well, that two comes because I have two terms, explicitly,

$$-\frac{1}{2}\omega^{ab}x_a\delta^c_b + \frac{1}{2}\omega^{ab}x_b\delta^c_a = -\omega^{ac}x_a = \begin{cases} -vx_3 & c = 0 \\ 0 & c = 1 \\ 0 & c = 2 \\ vx_0 & c = 3 \end{cases}$$

But this thing is the same thing as, raising the index, I get

$$\begin{cases} -vx^3 & c = 0 \\ 0 & c = 1 \\ 0 & c = 2 \\ -vx^0 & c = 3 \end{cases}$$

So $\delta x^a = \{\frac{1}{2}\omega^{bc}M_{bc}, x^a\}$

[Some confusion.]

In $SO(3)$, there are three angles, called the Euler angles, and this is a smooth manifold parameterized locally by those three angles. Whatever those parameters are, that's what I mean. When I take a variation of a vector, here's S^3, here's the identity, and then map over by a finite transformation and then bring it back. You'll get the action by the tangent vector at the identity to that thing.

Do I still need to show that if I plug in $\omega^{12} = \theta$ I get a rotation? Make that a homework. You should find $\delta x = y$ and $\delta y = -x$, which is a rotation around the z axis.

I was going to do this explicitly, but since we're running short on time, I'll leave it as a homework assignment. Let $J^k = \frac{1}{2}\epsilon^{kij} M_{ij}$, and you'll get $\{J^i, J^j\} = \#\epsilon^{ij}_k J^k = \#\epsilon^{ijk} J^k$. Usually you get

$$[J^i, J^j] = \underbrace{\#f^{ij}_k}_{\text{structure constants}} J^k$$

and in $SO(3)$ it turns out that $f^{ij}_k = \epsilon^{ij}_k$. Okay, so it turns out

$$\{M_{ab}, M_{cd}\}_{\text{P.B.}} = \eta_{ca} M_{bd} - \eta_{da} M_{bc} - \eta_{cb} M_{ad} + +\eta_{db} M_{ac} = f^{ef}_{ab,cd} M_{ef}$$

where these are now the structure constants for $SO(3,1)$.

Because we're short on time there are a lot of exercises today. The spacial part, each one of the structure constants should be contracted with an ϵ, so really we should be saying $\{J_{ij}, J_{mn}\} = f^{pq}_{ij,mn} J_{pq}$.

You may wonder about η. It exists because of the sign of $SO(3,1)$, if I replaced η with the Kroenecker δ we'd get $SO(4)$.

These things were written in momenta and coordinates. Any mathematician will tell you that you can recover the Lie group from the Lie algebra. I don't know what I should say first.

Basically, let me make the observation, if we only care about the Lie algebra structure, we can replace the Poisson bracket, where I plug in generators for a Lie group. There will be structure constants f that come from taking generators and plugging them into the Poisson bracket. I think the physicists and mathematicians agree on the concept of a representation. You take a Lie group, and then a representation is a group homomorphism, it acts on a vector space V, it would be a homomorphism $\rho : G \to GL(V)$. You usually talk about Lie algebra representations. In terms of the generators, our specific representation is in terms of a Poisson bracket and specific expressions for the generators, $\{\cdot,\cdot\}_{\text{P.B.}}, M_{ab} = x_a p_b - x_b p_a$. Then there were the Lie algebra relations $\{M, M\} = fM$. Now I want to look for a different representation where instead of the Poisson bracket I see the commutator $[\cdot,\cdot]$, and instead of functions on the phase space I use matrices. But I use the same structure constants $[M, M] = fM$.

So let $V = \mathbb{R}^{3,1}$ and I have (v^a) for $a = 0, 1, 2, 3$. Now $(M_{ab})^c_d \to (M_{ab})^{cd} \equiv \delta^c_a \delta^d_b - \delta^c_b \delta^d_a = (M_{ab})^{ce} \eta_{ed}$.

For homework show that $[M_{ab}, M_{cd}] = \eta_{ca} M_{ba} - \eta_{da} M_{bc} - \eta_{cb} M_{ad} + \eta_{db} M_{ac}$.

Next time I want to do the spinor representation and the field representation. For the field representation let $M_{ab} = x_a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a}$. Here V is C^∞ of some Minkowski space. Then you can compute this commutator.