Math Physics
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All right, so I want to finish this up today, so I'll try to buzz along. So remember we've been talking about electrostatics on $N = \mathbb{E}^3$. I turned this into a PDE course. If you've ever taken a physics class, electromagentism, it always turns into a PDE class. There are a couple of things you do initially. You want to solve Poisson's equation for the electric potential $\phi \in \Omega^0(N)$. This satisfies $\Delta\phi = *\rho$ for electric charge density $\rho \in \Omega^3_c(N)$.

Along the way, we've used Green's second formula to prove this lemma:

Lemma 1 *Let U be a connected open set with bounded closure; two continuous functions a, b from $\bar{U}$ harmonic on $U, \Delta a = \Delta b = 0$. If these guys satisfy the same boundary conditions, then they are equal on U.*

This gives us solutions to the Dirichlet boundary value problem.

For fixed $f \in C^\infty \delta U)$ solve $\Delta\phi = 0$ on U with $\phi|_{\delta U} = f$.

Such a solution, if it exists, is unique.

How do we solve this? So we want $\Delta\phi = 0$ in U. We won't have any charge sitting in U. Then the charge is sitting outside of U In order to find the field inside U we have to solve the Dirichlet problem. The field ϕ inside U doesn't, only depends on the charge distribution outside U up to the extent that it determines ϕ_0.

We've proved uniqueness in this roundabout way, using this lemma. The method also gives us solutions, so the method also implies uniqueness, and is a more "physical" approach. So that's what we'll get going on now. Recall Green's first formula, that for $a, b \in \Omega^0(U)$ we have

$$\langle da, db\rangle_U \int_{\delta U} a * db - \int_U a \wedge *\Delta b.$$

So we're going to use Green's first formula. Denote $C^\infty(U)$ as continuous functions on $\bar{U}$ that are smooth on U. I'll call $C_0^\infty(U)$ the elements a of $C^\infty(U)$ such that $a|_{\delta U} = 0$. Then $Harm(U)$ are elments $b \in C^\infty(U)$ that are harmonic on the interior.

I chose a and b to satisfy these conditions, and then Green's first formula tells us that these are orthogonal to one another. Then $dC_0^\infty \perp dHarm$ with respect to $\langle\ ,\ \rangle_U$

In particular, if $u \in C_0^\infty \cap Harm$, then u must be zero. Green's first formula implies more. That is, $dC^\infty = dC_0^\infty \oplus dHarm$. So what does this decomposition imply for us? For any $f \in C^\infty$, we can write $df = da + db$ uniquely for $a \in C_0^\infty$ and $b \in Harm$. Then there is a projection operator $dC^\infty \to dC_0^\infty$ with $\pi(df) = da$ and $(1-\pi)(df) = db$.

This gives us a way to solve the Dirichlet boundary value problem. Let me remind you what we want. We want Φ smooth on the interior and continuous on the closure, such that $\Phi|_{\delta U} = \phi$ and $\Delta\Phi = 0$. Take any $f \in C^\infty$ with $f|\delta U = \phi$. Take Φ as b up to a constant. With a very little bit of functional analysis we were able to solve this problem. To solve Poisson's equation you need more functional analysis.

One thing I want to point out is these minimization principles. You know that harmonic functions should minimize distance of one-forms. What does this have to do with harmonic forms with the boundary conditions here. Notice that

$$||df||_U^2 = ||da||_U^2 + ||d\Phi||_U^2 \geq ||d\Phi||_U^2.$$

Thus Φ minimizes the function $f \to ||df||_U^2$ among f that restrict to certain boundary values. This is step with what we know.

I'll just point out, we'll see that the modulus squared can be interpreted physically, so that $\frac{1}{2}||d\Phi||^2 = \frac{1}{2}||E||^2$, so in this sense what happens is that Φ minimizes the modulus squared, the energy.

Define $K(N \times N)\backslash\Delta \to \mathbb{R}$ via $K(x,y) = \frac{1}{4\pi}\frac{1}{||x-y||}$ for $x, y \in N$. I can write this for any two elements of N but it's only defined off the diagonal. For now fix x, and note that $*dK(x,\cdot) = \frac{1}{4\pi}\beta_x$, where this is the volume form for the two-form centered at x.

[Is that a β.]

Yeah, β, ahh, β.

[What's the joke?]

It's from the Simpsons. Everyone leaves and Snake, he robs the house, and he's stealing their VCR and they have a β

[–a betamax–]

and he says [laughing] "ahh, beta."

So onyway. The identity we derived last time was

$$a(x) = \frac{1}{4\pi}\int_{\delta U}(a\beta_a + \frac{*da}{r_x}) - \frac{1}{4\pi}\int_U \frac{*\Delta a}{r_x}.$$

So this is

$$\int_{\delta U}(a * dK(x,\cdot) + K(x,\cdot) * da) - \int_U *K(x,\cdot)\Delta a.$$

From the Dirichlet boundary problem, for fixed $x \in U$, the function $K(x, \cdot)$ is smooth on the boundary so I know there exists a unique harmonic function $h(x, \cdot)$ such that $h_U(x, y) = -K(x, y)$ for all y.

Lemma 2 *1. The function $h_U : \bar{U} \times \bar{U} \backslash \Delta \to \mathbb{R}$ is continuous and differentiable in y on U.*

2. For fixed x, $\Delta_y h_u = 0$.

3. $G_u(x, y) = K(x, y) + h_U(x, y) = 0$ for all $y \in \delta U$.

This is not hard to prove, but we'll see if we have time later.

I want to say some nice things about G. The second one,

Lemma 3 *$G(x, y) = G(y, x)$.*

Take a ball of radius ϵ centered at both x and y. Apply Green's second formula to U minus these balls. The functions we consider are $a = G(x, \cdot)$ and $b = G(y, \cdot)$. Now, both a and b are harmonic on the domain, and both vanish on th boundary δU. So we get

$$\int_{\delta B_\epsilon(x)} G(x, \cdot) {*} dG(y, \cdot) + \int_{\delta B_\epsilon(y)} G(x, \cdot) {*} dG(y, \cdot) = \int_{\delta B_\epsilon(x)} G(y, \cdot) {*} dG(x, \cdot) + \int_{\delta B_\epsilon(y)} G(y, \cdot) {*} dG(x, \cdot).$$

We claim that the left hand side goes to $G(x, y)$ as $\epsilon \to 0$ and by symmetry the right hand side goes to $G(y, x)$.

The second term of the left hand side, plugging in, is $\int_{\delta B_\epsilon(y)} G(x, \cdot) {*} dK(y, \cdot) + \int_{\delta B_\epsilon(y)} G(x, \cdot) {*} dh(y, \cdot)$. The second term here goes to zero, since these functions are both smooth on the whole ball. Then the first half gives $G(x, y)$ because $dK(y, \cdot)$ is β_y.

What about the other half? You can imagine that we're going to expand a G in terms of h and K. You get

$$\int_{\delta B_\epsilon(x)} K(x, \cdot) * dG(y, \cdot) + \int_{\delta B_\epsilon(x)} h(x, \cdot) * dG(y, \cdot).$$

The first term here is zero for the same reason as before. The first term is $\frac{1}{4\pi\epsilon} \int_{\delta B_\epsilon(x)} {*} dG(y, \cdot)$, which is

$$\frac{1}{4\pi\epsilon} \int_{B_\epsilon(x)} d * dG(y, \cdot).$$

Let's just kill this bird. G has these nice properties, it's harmonic one variable at a time, and so on. For any $a \in C^\infty(U)$, apply Green's second formula again, wher $b = G(x, \cdot)$. We'll do this on the domain $U \backslash B_\epsilon(x)$. So we have

$$\int_{\delta U} a * dG(x, \cdot) - \int_{\delta B_\epsilon(x)} a * dG(x, \cdot) + \int_{\delta B_\epsilon(x)} G(x, \cdot) * da = -\int_U G(x, \cdot) * \Delta a.$$

So expanding the left hand side I get

$$\int_{\delta B_\epsilon(x)} K(x,\cdot) * da + \int_{\delta B_\epsilon(x)} h(x,\cdot) * da = \frac{1}{4\pi\epsilon}\int_{\delta B_\epsilon(x)} *da + \int_{\delta B_\epsilon(x)} h(x,\cdot) * da.$$

The first term is of order ϵ and the second of order ϵ^2, so both of thsee go to zero. The other terms, well, we also have the $*dG$ terms, which gives us $a(x)$, and we're left with for any interior smooth, closure continuous function

$$a(x) = \int_U G(x,\cdot) * \Delta a + \int_{\delta U} a * dG(x,\cdot).$$

Using Green's functions, we can find the value of a smooth function at any point by looking at its Laplacian and its boundary values. So if $\Delta\phi = \rho_0$, and $\phi|_{\delta U} = 0$, then $\phi(x) = \int_U *G(x,y)\rho_0(y)$.

If you have G, you also have a way to solve the boundary value problem, so that if $\phi|_{\delta U} = f$ and $\Delta\phi = 0$ then $\phi(x) = \int_{\delta U} f(y) * dG(x,y)$.