Dennis Seminar
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Let's talk about the universal cover with a tiling and a presentation in terms of generators and relations for the fundamental group. I'll work in dimension two and then make a comment about dimension three.

[(whispered) Is he the frightening guy?]

[Lena: Yes.]

Take the surface, cut it open, is that enough? When you cut this open you get this eighteen sided figure, and look at the complement, you get this polygon, you lift the center and then lift the sides, and then take another lift, and you get a tiling, and so on. Now I want to draw pictures without [unintelligible]the size. You have, the fundamental group, since this is the universal cover, every loop opens up and you move one fundamental domain to another fundamental domain. So the elements of π_1 are in one to one correspondence with the tiles. This is not in a unique way, there are many ways to set up the correspondence. Pick one tile to be e, and then write the letter a for the path that moves to one of the adjacent tiles. So I'll label all the sides of the tiling. I have to relate that to the generators and relations in the fundamental group. Now $\bar{a}$ moves this second tile back to the first.

You have the covering space over the space, you have this region, you lift it up. I constructed it with paths, that was like the first or second day. The preimage of the cutting locus gives these figures, and then the complement has preimage these tiles. I want to relate the fundamental group to this picture. So suppose I have a tiling and a group of symmetries that has no mixed points, that moves the tiles to the tiles. A big element of the fundamental group will move this tile way over there, and another one way over there. I can move any tile to any other tile with an element and I want to decompose these into shorter pieces in the fundamental group. I start with the starting tile. Then I label the sides. So a means the group element moves this tile to this tile. Let's look at $\bar{a}$, it moves this one back to that one. So $\bar{a}$ moves it, say, to this tile. Just by applying it to the two things together. These things have even numbers of sides. So we go around and we label, and if an element appears, its inverse appears. This has geometric meaning in terms of translating to the tile you go across the edge of.

Now I'll take these labels, take a tile, take an element of the group that moves the one tile to the other, and then I move the labels over. Be careful, you need a mark, this one is the identity. So this tile might be "a''.

So you've done this now. You've labeled things. What is the group element that moves this tile over to this one? It's not a. I just moved the labels over according to what they were labeled by on the identity. So if you move back by $\bar{\omega}$ and then a and then back by ω you move across the side desired. So this is $\omega a \bar{\omega}$ or a^{ω}. So the a motion is conjugated by ω to give this other moton.

The fundamental group, in some sense, comprises the tiles. So now let's do, suppose you start somewhere and you do an a_1 and then you go across, you're in a new thing, you leav this tile by a_2, what is that movement? What group element does that? It's not a_2, it's $a_1 a_2 \bar{a}_1$. If you step by a_1 and then by a_2, first you do a_2 and then you do a_1, so it's backwards.

Let's do another example to see if it's a rule. You get $(a_1 a_2)(a_3)(a_1 a_2)^{-1}(a_1 a_2)$.

So it's $a_1 a_2 a_3$. So to make a walk with these labels, the group element that takes one tile to the next is to apply them in reverse order.

If you're standing here, which moves this tile back? It's $\bar{a}$. But if you conjugate, I was supposed to, well, with the labelling I should always have this picture.

If this group happened to have an element of order two, then a would be the same as $\bar{a}$.

So you have a directed graph which is labeled with the group elements. If you took your walk and wrote it down from left to right, then the group element that moves one fundamental domain to the next is the product, taken as operators, so in the reverse order.

If you have two ways to walk to the same place, it means that the two ways to walk there are the same, so you have a relation. This seems like an invention, no, it's an actual statement so it's a proposition. If you walk along

[How come when we talk about topology of surfaces we always get a geometric interpretation?]

It can in dimensions two and three, not in higher dimensions. This argument is purely topological, I was doing the geometry to get the tilings. The proposition is that when you take a walk like this, this is how the group theory works.

Now, so we have this graph associated to this figure. This is called the Cayley graph of the group.

[I was thinking about this question, I don't see a lot of geometry there.]

The universal cover, yeah, right, I don't need the geometry, I just get examples from that. Whenever you have a space you can cut it and you get a manifold with boundary, and the whole universal cover is tiled by copies of that manifold with boundary, and each one of the pieces of boundary corresponds to a generator, and now note that, a second proposition, any group element is a word in the initial group elements. I can walk to any tile, and then this tells me how to write the group element down like that, and this is showing how to get to any tile by taking a word in those first letters I wrote down. The first set of letters are group elements, the ones further out are conjugated group elements, but these all allow you to get from one tile to any other tile.

This is called a symmetric set of generators. You only need to use exponent $+1$ to get them. You normally need to explicitly include inverses. So in this, any element is a positive word in these basic elements. This is called a symmetric generating set. I'm throwing that in because the way a generating set works usually, you have to close by inverses.

Now, if you think of the dual picture, where I put a point in the middle of each cell and draw lines across each of the walls, and so on, that gives me a graph associated to this, and, uh, it's conceivable, for example, if you took a surface with boundary and did this, then the graph would be a tree. You could make fundamental domains by breaking this into two hexagons. We don't have boundary, so we don't have a free group, and this thing is obviously not a tree. This is a famous structure due to Poincaré, the dual cell decomposition which is made out of triangles because you started with a trivalent graph. So we can think then of the group G as the set of words in our alphabet $\{a, \bar{a}, b, \bar{b}, \ldots\}$, but I have to take words grouped into equivalence classes. We think of a word, if we're reading from left to right, if we have two paths that end at the same point, that completely determines the elements of the group. We can group these into classes by saying two words are equivalent if and only if their paths end at the same point. You can think you're walking in this graph. If you can get from one to another, this is a discretization, [unintelligible].

This is finitely many generators, infinitely many relations, but these are generated by a smaller number of relations. There are ways to take a smaller set of relations, and it turns out there are really basic relations. All of these relations are generated by what you might call the vertex relations. I need a nicer picture. I erased my example. That's just, if you have these, if you have this path, and then you have another path, you can, we're in the universal cover, which is simply connected. You know you can deform one path to another. If you mave a little bit, you can replace one by two and so on. I can move a global path to a global path. The moves will move across at most one vertex. There's a relation associated to a given vertex, and you're just sticking on other consequences of the relation. You can think of the path as, in this way, take the group elements that do these things. It says if you move in a little triangle, that's a relation. You can write this as $abc = 1$. But if, back in the original fundamental domain, I've got a triangle too, that will be mapped to this triangle by mapping these three things to here. You, this is the one I started with. This is just a copy of that, this whole picture is the same over there. These transformations are the conjugates of these transformations of the original fundamental domain. So in fact, the vertex relations are all conjugate to vertex relations at the starting tile. We can stop here with a finite number of generators and a finite number of relations.

These relations themselves are not independent. Sometimes you can map, these vertices may be related by group elements. So if the two vertices are related by the group, you just write that out. You have an 18-gon. We have eighteen generators, nine pairs, and then at first glance we'd have 18 triple relations, but when you look at the quotient, there are only 6 independent vertices. So the three relations are all conjugate at each vertex. So you have 6 independent vertex relations. This is a presentation that has 9 generators and 6 relations. That means there are kind of three independent generators. Another way to do it is with four generators and one relation, which gives you the same number. In fact, what you end up with is, for surfaces you have two cases. For the closed case, I'm doing the description. It turns out that if you squeeze all your things to a point, you get four generators and one relation, the relation is longer because you have to go around more edges. For closed surfaces you get a free group of genus g, say,

Hello? 4:30, okay.

So you finally get four generators and one long relation. This presentation is not actually as efficient. One knows that by Abelianizing. In other words, it's almost free. If it's not a closed surface, then π_1 is actually free. When it has boundary, you can shrink it down to a graph, so the universal cover is free.

There was a big theorem back in the day, a subgroup of a free group is free. A free group would be a graph, a subgroup would be a covering of a graph, and the universal cover would still be a tree.

What's complicated about these groups is that the automorphism groups of the free groups are very rich. I could name at least four different parts of mathematics where all the unsolved problems are related to those automorphism groups.

So you take these Γ and then you want to find them as discrete groups of non-Euclidean isometries.

Just choose a matrix for each generator, if it's free. In the closed case you need your matrices to satisfy this one relation. The situation is, looks kind of simple but it's not.

But going back to the cute fact about three dimensions, in terms of abstract group theory, we only got free groups and one-relation groups, so we haven't really probed group theory.

The way the group sits on the surface is a complicated thing. When you have groups like this. You're studying curves with a basepoint up to deformation, and sometimes you let the basepoint go. This defines a conjugacy class of elements.

In the free group there's a nice picture of conjugacy classes, you write your word in a little circle. You can move the d from the end back to the beginning. So you write the word in a ring, and they're all conjugate. So closed curves in the surface that is not closed correspond to rings in the generators up to conjugacy. There's a branch of mathematics called combinatorial group theory, Linden [unintelligible],

Group theory is a bad branch of math, because all of the good questions are not decidable.

Suppose you know a group is nontrivial. There is no computer program that will tell you about the equivalence of two words, in general. In certain groups all these questions have answers. So this can be done for two and three dimensions (now thanks to Perelman). The groups in four-manifolds include all finitely presented groups, so everything is undecidable.

Naw take a 3-dimensianl space. Take a solid thing and pack them together into what is called not a tiling but a chain ring. Then there's a group that moves these around. We're looking at the universal cover af a three-manifold. You label the sides, and then get the same thing with paths. Now, since the universal cover is simply conncted, you can move one path to another. You only have to cross edges. So then when you move across an edge, you get a vertex relation. You can go around the edge, you get a similar picture, it has one generator for each face and one relation for each edge. You have the Poincaré dual decomposition. Let me imagine you have a bunch of these chambers. You have one node for each chamber and one edge for each face. You'd get a chambering, and then the dual chamber. The faces correspond to the edges and edges to faces. So you have two dual presentations of the fundamental group. So the numbers of generators and relations interchange, so there's some duality. This only works in dimension three. You can define an invariant of a group, something like the infinmum of the number of generators, maybe supremum, of the number of generators minus the number of relations.