Dennis Seminar
Stavros Garoufalidis
Resurgence in Quantum Topology
September 27, 2006

Gabriel C. Drummond-Cole

September 28, 2006

Also a gauge theory meaning that the trivalent graphs represent tensors coming out of Lie algebras.

I want to start with some examples of power series and then tell you where they come from.

[Why do power series relate to quantum theory?]

Perturbative field theory always gives you some grading, then you collect coefficients and regularize them to make things finite. There are too many graphs for the series to be convergent.

[These power series are supposed to be part of some global holomorphic object.]

I want to talk about how to construct in a canonical and computable way a holomorphic object out of a divergent power series.

Let me start with some examples. First a definition before I do examples.

Definition 1

$$F(x) = \sum_{n=0}^{\infty} a_n \frac{1}{x^n}$$

is Gevrey-s, if $|a_n| < C^n (n!)^s$.

So F is Gevrey-0 if and only if $F(x)$ is convergent for x large enough. There is no good characterization of Gevrey-1 series. One example is $F(x) = \sum_{n=0}^{\infty} \frac{n!}{x^{n+1}}$. This doesn't just come out of a hat, I claim $F(x)$ satisfies the following ODE due to Euler, $F'(x) + F(x) = \frac{1}{x}$.

To prove this, differentiate to get

$$F'(x) = \sum_{n=0}^{\infty} -n!(n+1)\frac{1}{x^{n+2}} = -\sum_{n=0}^{i} nfty(n+1)!\frac{1}{x^{n+2}} = -\sum_{n=1}^{\infty} n!\frac{1}{x^{n+1}} = -(F(x)-\frac{1}{x}) = -F(x)+\frac{1}{x}.$$

So in fact $F(x)$ is the unique formal power series solution to this ODE. Any ODE will give you a fixed point algorithm to find the unique solution. You want a functor from formal power series to ordinary functions which commutes with addition, multiplication, differentiation, integration.

If you plug in, say, .01, then you will have your error going down, and then eventually up exponentially.

[Like fighting with your wife.]

This is called the least term truncation.

Anyway, this is a baby case, in our situation, there won't be a differential equation.

To construct the function, getting a little ahead of ourselves, there is something called the Borel transform $\mathbb{C}[[\frac{1}{x}]] \to \mathbb{C}[[p]]$, which takes $\frac{1}{x^{n+1}} \mapsto \frac{p^n}{n!}$, so that

$$F(x) \mapsto G(p) \mapsto \mathscr{F}(x) = \int_0^{\infty} e^{-xp} G(p) dp,$$

where this last step is the Laplace transform. Now if $F(x)$ is convergent, then $\mathscr{F}(x) = F(x)$ and $G(p)$ is entire of exponential growth, so that there is a Laplace transform, and you can get $\int_0^{\infty} e^{-xp} \frac{p^n}{n!} dp = \frac{1}{x^{n+1}}$.

You need class one because you want the Borel transform to give you something convergent in a neighborhood of zero.

In our baby example you have

$$G(p) = \sum_{n=0}^{\infty} \frac{n!}{n!} p^n = \frac{1}{1-p},$$

which is valid for $|p| < 1$, and can be analytically continued on $\mathbb{C} - \{1\}$. Now the integral

$$\mathscr{F}(x) = \int_0^{\infty} /e^{-xp} \frac{dp}{1-p},$$

which is well defined for $Re(x) > 0$, and $\mathscr{F}' + \mathscr{F} = \frac{1}{x}$.

This is a divergent, a formal power series that converges on a half-plane. If you wish, $\mathscr{F}$ is asymptotic to the original series for large enough x. This means

$$\left| \mathscr{F}(x) - \sum_{n=0}^{N-1} \frac{n!}{x^{n+1}} \right| < \frac{C_N}{|x|^{N+1}}$$

for $Re(x) > D_N$ and all N. This has an essential singularity at infinity, but there is somewhere around which it converges.

This is the simplest case, but now the kind of analysis we're going to be doing, we won't be using differential equations at all.

[Do you know what Borel was considering?]

No. There is some discussion of this in Hardy, "Divergent series."

[When did Ramanujan write to Hardy?]

Before this.

How about an example that is less, that is closer to home. Let me write

$$F_{3_1}(x) = e^{-1/24x} \sum_{n=0}^{i} nfty(1-e^{-1/x}) \dots (1-e^{-n}x) = \sum_{n=0}^{\infty} (q)_n \Big|_{q=e^{-1/x}}$$

where $(q)_n = (1-q)(1-q^2)\dots(1-q^n)$ is the quantum n-factorial. Here the coefficients of, say, $\frac{1}{x^n}$ will depend on the first n terms.

If you wanted to sum $n!$ from zero to infinity, which is even more exciting than summing n from zero to infinity.

This is called F_{3_1} because it is coming from the simplest nontrivial knot, the trefoil. So F_{3_1} is Gevrey-1, and we can define the Borel transform $G_{3_1}(p)$ which is convergent for small enough $p \sim 0$.

I didn't mention my collaborators, Le and Costin. With Costin we get

Theorem 1

$$G_{3_1}(p) = 54\sqrt{p}\pi \sum_{n=1}^{\infty} \frac{\chi(n)n}{-6p + n^2\pi^2}^{5/2}$$

where $\chi(n) = \begin{cases} 1 & n \cong 1, 11 \mod 12 \\ -1 & n \cong 5, 7 \mod 12 \\ 0 & \textit{otherwise} \end{cases}$

So singularities of $G_{3_1}(p)$ are at $\frac{n^2\pi^2}{6}$ for $n = 1, 5, 7, 11 \mod 12$. These are thus not equally spaced. You don't need to to do anything funny if you start with a formula like the one we used, you can start with maple or mathematica. So then this $F(x)$ turns out to be $1 + \frac{23}{24}\frac{1}{x} + \frac{1681}{1152}\frac{1}{x^2} + \dots$. Then $G(p)$ becomes $\frac{23}{24} + \frac{1681}{1152}p + \dots$ So I'd better get that in my formula.

So at $p = 0$ we get

$$54\sqrt{3}\pi \sum_{n=1}^{\infty} \frac{\chi(n)n}{n^5\pi^5} = \frac{54\sqrt{3}}{\pi^4} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^4}$$

$$= \frac{54\sqrt{3}}{\pi^4} L(x, 4),$$

where $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^5}$ is an L-function, a cousin of $\zeta(s)$; and $\zeta(2n) = \pi^{2n}$ for a rational number.

So $L(x, 4)$ can be independently computed and you can check that $\frac{54\sqrt{3}}{\pi^4} L(\chi, 4) = \frac{23}{24}$.

If you wanted to know the first derivative at zero, you would differentiate and then set $p = 0$, which would give you a term which is $L(\chi, 6)$ plus some lower order terms with some $L(\chi, 4)$.

So this is the first interesting knot. The next simplest knot is the figure eight, which has a power series $F_{4_1} = (q)_n (q^{-1})_n|_{q=e^{-1/x}} = \sum_{n=0}^{\infty} (1 - e^{-1/x}) \ldots (1 - e^{-n/x})(1 - e^{1/x}) \ldots (1 - e^{n/x})$.

We don't have a nice expression for $G_{4_1}(p)$, but we know that it has an analytic continuation on $\mathbb{C} - \lambda\mathbb{Z}*$, where $\lambda = i\mathrm{vol}(4_1) = i(2.02\ldots)$

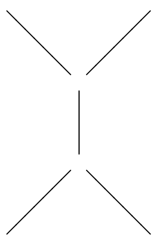
This is one of the two simplest hyperbolic three-manifolds, the complement of the figure eight and its sister.

So where do these power series come from, and how do we generate them? Now I want to tell you how to generate power series from perturbative quantum field theory.

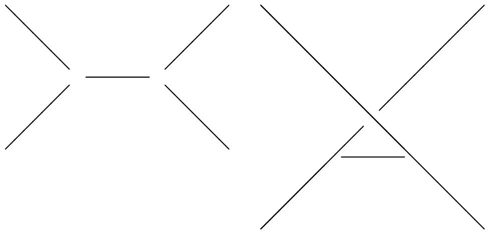
I hope I convinced you that these are discrete power series, they are not abstract. I should have stated that there is no differential equation, linear or not, with polynomial coefficients, which has that as its solution. So resurgence (which I have not defined) is not obvious.

Okay, so how do I generate power series from perturbative three-dimensional quantum field theory.

There are two types of knotted objects, either knots in 3-space or closed 3-manifolds. So I have some 3-manifold M, and then there is a graph-valued invariant $Z(M)$. These are power series whose coefficients are trivalent graphs. The graph-valued invariant lives in $\mathscr{S}(\emptyset)$ which is series of trivalent graphs with vertex orientations modulo two relations: changing an orientation changes the sign, and the three term relation



is the difference of the following two graphs



These are cubic Feynmen diagrams up to Lie algebra relations. The degree of a graph is half the number of vertices. Then $\mathscr{S}_+(\emptyset)$ is generated by the two graphs Θ and $\bigcirc$ —— $\bigcirc$ modulo these relations. Then Z_M is supposed to be the perturbation expansion of the Chern-Simons path integral along the trivial flat connection.

So from the graph valued invariant by means of the weight system from a simple Lie algebra $\mathfrak{g}$ we can get something in $\mathbb{Q}[[\frac{1}{x}]]$ and so $Z_{\mathfrak{g},M} \in \mathbb{Q}[[\frac{1}{x}]]$. We want to prove that these are all Gevrey-1, and with the least possible pain, for all M and $\mathfrak{g}$.

First I'll explain and then I'll prove. First let's explain. Well, $\mathscr{S}_n(\emptyset) = \mathscr{D}_n(\emptyset)$ modulo relations where $\mathscr{D}_n(\emptyset)$ is a $\mathbb{Q}$-vector space with basis trivalent graphs with $2n$ vertices.

Lemma 1 *the dimension of $\mathscr{D}_n(\emptyset) \sim (n!)^3C^n$. You start with a trivalent graph, cut along edges, and glue the cuts. This gives you $2n$ copies of a Y. These have $6n$ legs. So the gluings, there are $(6n-1)(6n-3)\ldots = (6n-1)!! \sim (3n)!C^n \sim (n!)^3C^n$. Bollobas considered random trivalent graphs and showed that there are no isomorphisms between them, so that this bound is asymptotically correct.*

This gives Gevret-3, not Gevret-1. If you divide by the wrong factor you won't get any singularities, be entire, or you won't converge at all near zero.

So the right powor of $n!$ is one and not three, where is it? It comes out of the relations. Start with a big graph.

I can use my relations to destroy a bunch of the vertices, and you can make things into a chord diagram. Since to begin with you had $2n$ vertices, now you have n chords. You have paired vertices. So that's $2n!! \sim n!C^n$. You might be smaller. No one knows.

We know that $\sqrt{n!}C^n \leq \dim \mathscr{S}_n(\emptyset) \leq n!C^n$.

These series may not be Gevret-1, we also need to know that the coefficients are exponential at worst. In order, now, a proof that works for all Lie algebras and beyond, requires one more idea, the Gromov norm.

The Gromov norm o graph valued invariant is as follows. Suppose that V is a vectorspace spanned by v_j for $j \in J$. If $v \in V$ then $|v|$ is $\min\{\sum |c_j| = \sum c_j v_j\}$. Then $|\ |$ is a norm. In particular $|v| = 0$ if and only if $v = 0$. So apply it to $Z_M = \sum_n Z_{M,n}$ for $Z_{M,n} \in \mathscr{S}_n(\emptyset)$, trivalent graphs with $2n$ vertices. Then $|Z_{M,n}| \in \mathbb{Q}$ annd $|Z_M| = \sum_{n=0}^{\infty} |Z_{M,n}| \frac{1}{x^n} \in \mathbb{Q}[[\frac{1}{x}]]$.

Theorem 2 *(Le, G.)*

1. *For every three-manifold,* $|Z_M|$ *is Gevrey-1.*
2. *For every simple Lie algebra,* $Z_{\mathfrak{g},M} \in \mathbb{Q}[[\frac{1}{x}]]$.

Now I have to tell you how to compute this invariant. I'm only going to sketch it, because being off by an exponential in n is not a big deal. So let's do the LMO invariant of M, also called the Aarhus integral.

So start with a surgery presentation of M as surgery on a framed link in 3-space. You can cut, twist and glue, if you did have a link in three-space. Then you consider another graph-valued invariant, the Kontsevich integral, Z_L.

[Anarchy as Dennis leaves for a phone call.]

A specific manifold has more than one presentation as a surgery. Then you consider the Kontsevich integral, which goes over trivalent graphs with legs colored by the components of the link. So then using the framing, you glue the legs of the same color to each other.

To prove that something is Gevrey-1, you have to understand this process, but only from a distance.

What is the Kontsevich integral? First, what is a knot? It's a finite object assembled together from a few elementary blocks, and the Kontsevich integral is local, meaning it's based on specific pieces of the knot, sliced up into these blocks.

Lemma 2 *The product of Gevrey-s series (rather, their inverses) is Gevrey-s.*

So the Kontsevich integral assigns some things like $exp(\sum b_n \Gamma)$, Φ (Gevrey-0), and $exp(\frac{1}{2}|\dots|)$.

Theorem 3 $|Z_L|$ *is Gevrey-0.*

I use an associator on one part of this. The bad news is that, associators themselves have Gromov norms. These are series in graphs on three vertical strands. There are associators with arbitrarily large Gromov norms. There is one, only one that we know, that is Gevrey-0, because of an explicit formula.

Okay, so what's the story? You start with knotted objects M, move to a Gevrey-1 formal power series $F_M(x) \in \mathbb{Q}[[\frac{1}{x}]]$, and then use the Borel transform $G_M(p) \in \mathbb{Q}[[p]]$, which is convergent for $p \sim 0$ and enless analytic on $\mathbb{C} - Nn$.

Conjecture 1 *$\mathscr{N} \subset \mathbb{N}\{i\mathrm{vol}(\rho) + CS(\rho)\}$ whre ρ is a parabolic $\mathfrak{sl}_2(\mathbb{C})$ representation of M.*

The moduli space of $\mathfrak{sl}_2\mathbb{C}$ rpresentations $i\mathrm{vol}+CS$ is constant on each connected component.

[missing.]

Let's look at $y' = y + \frac{1}{x} + y^2$. Then the solution is $y(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots$, whose Borel transform has singularites at $1, -1, -2, -3, -4, \ldots$ (Ecalle)

Suppose we're given a power series $G(p) = \sum a_n p^n$ which is convergent for small enough p. We say that G has endless analytic continuation if for every positive L (length) there exists a finite set in $\mathscr{N}_L$ so that for every path γ that starts at 0 and has length L and avoids $\mathscr{N}_L$, G has analytic continuation along that path.

Let me give you a slightly nontrivial example. Take $G(p)$ to be the dilogarithm $\frac{p^n}{n^2}$. This is convergent for $|p| < 1$. Why does it have endless analytic continuation? I can write it as $\int_0^p \frac{\log z}{1-z} dz$. So by this formula it has analytic continuation on $\mathbb{C} - [1, \infty)$, with a jump along the cut of $2\pi i \log z$. You pick up a $(2\pi i)^2$ going past the other cut. This proves that this function is a multivalued function on $\mathbb{C} - \{0, 1\}$. Some explanation for why the second cut is needed.

A homework problem of use to what we are doing, is to show that $\sum \frac{1}{\sqrt{n}} p^n$ is resurgent, that is, has endless analytic continuation, is multivalued in $\mathbb{C} - \{0, 1\}$. However, $\sum p^{n^2}$ is not resurgent. You don't have to use weird examples to get non-resurgent functions.

[Why do you make this conjecture?]

I didn't say. The analogue of the differential equation is the finiteness of the knotted object.

1 Kate Poirier

I don't know if anyone else has looked at chapter 13 yet. I can tell you what the contents are. Are you typing? It's on the internet already.

13.1 These are examples of quotient spaces. These should look like Euclidean spaces modulo a group action. I thought we could spend some time trying to understand these examples.

13.2 This is basic definitions, which maybe we can talk about.

13.3 Maybe next time we can discuss the beginning of classifications of 2-dimensional orbifolds.

So the first example is $\mathbb{R}^3/\mathbb{Z}_2$, where we take the half-space.

[Is it important that the points of the action which are non-manifold points are fixed?]

[They are organized by which subgroups fix them.]

The next example is $\mathbb{R}^3/zZ_2 * zZ_2$. This is going to be generated by reflections in two parallel planes. This is the barbershop model, when you have mirrors on both sides.

[What does everyone think that group is? How big is it? The even length words are the translations, that's the integers, and the quotient is the circle. Then the further finite group of symmetries gives you the interval.]

The next example involves this too, it's $\mathbb{R}^2/D_\infty \times D_\infty$. You get a rectangle.

The next example I start having trouble. Consider only the subgroup of index two which preserves orientation.

To start to figure out what the quotient space is, he draws the picture of the two adjacent rectangles, and then figures out what is going to be identified.

Here's the last example, the hardest. Look at a lattice in $\mathbb{R}^3$, and consider the lines bisecting the edges, not intersecting. Rotations of π about these bisectors generate the group.