Dennis Seminar
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We were describing the, we were looking at these nontrivial covering spaces, we were talking about Thurston, trying to communicate it in an exam, and his communication skills weren't enough to convince the examiners that he understood. That's understandable, this is complicated. So what I've done, I'm looking at the surface of genus two, and the generic way of cutting it so that the complement is connected and it's a cell. When you cut it open you get an 18-gon. At every vertex there are three sectors. You imagine pulling this thing tight, it's made of string, if you really pulled it tight you'd have 120 degrees at every corner, and then you have to fit three of them together at every vertex. The universal cover exists, each one of the nine sides appears twice in this pattern. You can think of it as, take these rays to the vertices and lift to the universal cover, and then you have another copy next to it, and this would be on another sheet. [Pictures]. Then you fit another one in here. Somehow the universal cover is tiled by these eighteen-gons, three fitting together at a corner, [unintelligible]sitting on the floor here.

So what is this incredibly huge thing we get made out of so many pieces? It looks like when you go around each one has eighteen around it.

We tried to get a geometric picture of this, look at geometric pieces. Looking at an eighteen-gon in the plane, the internal angles would be quite large, and so you can't fit together three of them in the plane, it curls up out of the plane and you get a very crumpled thing, twisted, but abstractly it exists. We constructed the universal cover of the surface. The question, consider the curve, take two points on either side. Every arc to connect two such things is essential, can't be squeezed to a point, and then the two arcs are seperated, embedded.

So this is not how these things were discovered. This whole construction of the fundamental group came around 1900 and a more geometric understanding of this example came 15 or 20 years earlier using differential equations in the complex domain, complex analysis, and non-Euclidean geometry, all of which were in play in the 1800s.

These examples were so nontrivial that they forced Poincaré to start with covering spaces, the fundamental group, and so on. There is this non-Euclidean geometry, this infinite thing like the plane, it has geodesics and things like that, and there are lots of geodesics that don't cross a given one, you can have whole families. When you have two geodesics that don't cross, generically there is a unique common perpendicular. You can make it move off to infinity. So there are asymptotic geodesics where the common perpendictular is at infinity. So this, the full history of this depends on how you talk about it, but it was worked on by Steiner, Lobachevski, Gauss, it was kind of like Copernicus, kind of heretical. Mathematicians who used it were out on the fringe. Gauss was discrete. Even Poincaré would convert arguments back into complex analysis at the beginning, eventually he got headstrong.

Most surfaces we're talking about are non-Euclidean. You can take a point and draw an 18-gon near the point, and it would be almost like a Euclidean 18-gon, with angles very obtuse. Then you could pull the vertices off to infinity to get bigger 18-gons. Eventually in the limit it consists of 18 asymptotic geodesics. When they get very far away, the angle between them is very small, the area stops changing. It's like 16π or something like that. Anyway, that shows that when it's very small you have a very flat angle, more than 120 degrees. That's the hexagon. The heptagon has more than 120, and so then when it's big the angle is less than 120, so there's a place in the middle where all the angles are 120 degrees, by this continuity argument. This works pretty well in general, this argument, in hyperbolic space. Now we can take this piece of non-Euclidean geometry.

Anyone here want to know what non-Euclidean geometry is about? Before Thurston it wasn't in mathematics. Then sort of he discovered it in high school. You can make non-Euclidean paper.

Exercise 1 *Make non-Euclidean paper. I'll explain horocycles. I asked Thurston, what horocycles are? He said draw the circle through me with center him, and he started to walk away. So the bigger and bigger circles are passing through me. If you do this in the plane, these circles limit to a straight line. They always have a definite curvature, greater than one, in the hyperbolic plane, and in the limit the curvature is one. There are a lot of parallel horocycles.*

[How is this in the disk model?]

Don't use those models, that's my gift to you, you have to live in the space, it's infinite.

[Have you seen the television program where Colin Rourke goes into hyperbolic space?]

No, but that reminds me of something. There's an ordinary Hollywood movie where a guy you're not supposed to like is in traction in the hospital, and he's putting on the TV, and he drops the remote and he got stuck on the free university, Colin Rourke teaching a class about something.

Cut a bunch of concentric circles, well, arcs, really thin, radii close to one another and glue them together, this to this and so on. To really be impressed, you make it and you get something like a pringles potato chip. Make two of them and you can put one on top of the other and you can fit them as they turn. A lot of these to make a rainbow.

I forget the practicalities of doing this. You have to do some shifting to get a nice piece. This is based on the idea of horocycles. Take an equivalence class of asymptotic geodesics. They form a fan and the orthogonal trajectories form the horocycles.

Now let's go back to the discussion. You have this infinite piece of hyperbolic paper, and I think Hilbert showed that you can't embed this in Euclidean space.

Now you can actually fit the three together, like hexagons in the plane. You can actually tile your hyperbolic bathroom floor with these 18-gons, and you have a big group acting on this. They didn't quite have this presentation, but they had slightly different things, where they had this symmetry, if you divide by the symmetry of the surface of genus two, you get a sphere with six branch points, and you can compose this with the universal cover. This is equivalent to the upper half plane or the unit disk, so we have a holomorphic map of the upper half-plane into this space. Then there are differential equations satisfied as you move around the regular singular points. There was a study of this by Poincaré and Fuchs.

[What's a branch point?]

Consider $z \mapsto z^2$. This maps the origin to the origin and the circle around the origin goes around twice. Geometrically, pulling back the geometry, you see a two story parking garage, and a funny point, a branch point.

Now you can perturb your 18-gon, keeping certain sides identified. They're identified by the topological picture. Divide by the group, it's locally modeled on the non-Euclidean geometry. It gives you a compact surface with non-Euclidean geometry.

There are nine lengths here, we also did the surface of genus three, we actually counted this and we made things generic and got 15 sides. We got 15 and you open it up and get thirty. The number of holes is called the genus, and in general it's $12g - 6$. The generalization of the picture gives you $12g - 6$ sides and they all have 120 degree angles. I'm going to stick with the 18 case, if we get anything I'll make it more general. It turns out that the geodesics adjacent to the same edge, at $120°$, don't intersect, we can perturb this slightly, we almost have a regular 18-gon. Don't put in the last three sides. Change the lengths arbitrarily to be near the old length. Then the other sides will go off, and as you move down the common perpendicular the angle gets bigger, so the last three side lengths are predetermined.

The whole point is that the last three sides are determined, fixed, by the first few. Fix one side with a length near one, and then after 15 the other three side lengths are determined. There's a unique way to complete the figure. When you have this n-gon with given angles you want the angles fixed, and now you perturb the lengths, and then you can perturb all but three. That's the proposition. This may be a little fast of a discussion. If I have the surface of genus g, which only gives half as many fixed lengths, so I get $6g - 3$, which is how many lengths we have that are obviously free, but then the last three we have no control over, so we get $6g - 6$.

Exercise 2 *Figure out what to do about the three sides that you have no control over.*

You can talk about the space of geometries on the particular non-Euclidean compact manifolds, and it's a nice manifold of dimension $6g-6$.

Now what is this non-Euclidean geometry? Some of these constructions can be connected to complex analysis and you can construct a bunch of nice functions. Then Poincaré made it a purely topological theory. That covering spaces are purely topological is genius level.

[Is that like de Rham's theorem before its time?]

You have to go back to Riemann to draw that connection. Like $\log\ z$ is an Abelian cover, you have to talk about Abelian covers. Riemann talked about it that way. That's like a first order differential equation. Second order differential equations gave much fancier covers.

So what is, we have this theorem started by Riemann about the universal cover of a connected Riemann surface is either the whole plane or the upper half-plane, and also the unit disk. This representation is a simply connected Riemann surface equivalent to the upper half-plane via $z \mapsto \frac{z-i}{z+i}$. In the disk there is a nice rotational symmetry that fixes the origin, $e^{i\theta}$, and that's the derivative. Then the derivative will be the same when you transport that to the upper half plane. These are one by one matrices so they change by conjugation, i.e., stay fixed. Then there's a metric, well, $ds = \sqrt{dx_1^2 + dx_2^2 + \cdots}$ and integrating along a curve gives the length. So $ds_{\text{non-Euclidean}} = \frac{ds_{\text{Euclidean}}}{y}$. The proposition is that $ds_{\text{non-Euclidean}}$ is invariant under all holomorphic bijections of the upper half-plane, which connects holomorphic stuff with geometry. These are $z \mapsto \frac{az+b}{cz+d}$, and these are real with $ad-bc=1$, or positive, it doesn't matter. This is called $Sl(2,\mathbb{R})$ and the quotient is $PSl(2,\mathbb{R})$. It's a three dimensional Lie group that contains the affine group and also the rotation around i. Translations and dilations don't affect the quotient in $ds_{\text{non-Euclidean}}$, and then you can restrict to those that preserve i.

Exercise 3 *Show by plugging in that* $(\frac{ds_{Euclidean}}{y})^2$ *is invariant under* $z \mapsto \frac{az+b}{cz+d}$ *for* a,b,c,d *real. It's nontrivial that this is invariant under the rotation.*

This metric, which is completely invariant, is a metrical representation of non-Euclidean geometry, so you can move this all over to the disk too. If you draw a shape, it's congruent to something which is the same size when you scale down.

That's the non-Euclidean metric. The holomorphic transformations are the same as the ones of non-Euclidean geometry. So any time the universal cover is the disk or the upper half-plane, all others carry a natural non-Euclidean geometry, since the deck transformations upstairs preserve it. You end up with this picture that every Riemann surface is either the round sphere, the flat plane, infinite cylinder, torus, or has a complete non-Euclidean geometry. Every topological surface is a smooth and Riemannian surface, and then a Riemann surface.

We'll discuss consequences and so on, maybe a little more of this.