Dennis Seminar
September 14, 2006

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From now on, I'll try to get here on time, and at the beginning talk about the things I got wrong. So first, the compact open topology. A topology is a collection of subsets, called open set, which includes the empty set and the whole space, and which is closed under unions and finite intersections. A basis is a collection which generates the open sets under union. A subbasis generates the open sets under first finite intersections and then unions.

There's an interesting topology on the space of all maps between two topological spaces. Take a compact subset K of the domain and an open set in the range, and maps that take the one into the other, that forms a subbasis.

Another thing I wanted to modify slightly is the definition of path. Let's define a path to be, I was inspired by the physics course this morning, it's a map $\mathbb{R} \to X$ which is eventually constant near $\pm\infty$. I won't write this down, but it means it's a map of the real line where all of the movement happens in a compact set. You identify two via translation. The whole point of that device is that when you compose two paths, you translate the action that started where the other one stops, and you put the paths together. You don't rescale to be the unit interval. When you compose with rescaling you get a nonassociative product. You could make this fancier, say a path is a map of a Riemannian one-manifold with boundary into the space, up to isometry. I was going to remark, it's a good thing this More idea wasn't around when Stasheff did his work. He constructed homotopies between the different ways of composing. The structures of this are very interesting in their own right.

[Is that A^∞?]

Yeah, I didn't want to use any words. There's interesting structure in what I'm sweeping under the rug, but I don't want to deal with that for now.

I want this to be a little bit painful because I wanted you to see that you can get more than just covering spaces out of it, you can get bundles with connections. So let's go back over it in the more usual way. Let X be a space and consider paths from the basepoint to itself. Call two equivalent when there is a deformation of one to the other keeping the basepoint fixed. This has a group structure under path composition with inverse reversal of direction and identity the constant path. We call this $\pi_1(X)$. So we can consider a subgroup H of $\pi_1(X)$ closed under composition and inverse. We'll say that paths $\alpha \sim \beta$ if they end at the same point. and $\bar{\beta}\alpha \in H$. So then we can form, given $x \in X$, you put over it the set of equivalence classes of paths that go from $*$ to x. Then we had this assumption that every point in X has a neighborhood which is path connected, and a smaller neighborhood which is simply connected.

The most important property is, take one of these neighborhoods and take another point, and you have the equivalence classes coming to one and to another. You can move the equivalence classes over, one to another. It preserves equivalence classes. It also, if you chose another path to define this map, you would get the same map between equivalence classes since you can fill in between two paths. So that means that there is a unique correspondence between the two fibers, a canonical one. So this gives you the famous path-lifting property. Say you had a path between any two points. You have neighborhoods with this property covering the path, so you can move uniquely along the path between the equivalence classes of paths. This is called the holonomy. Some people only call it holonomy when you go around a closed loop. If you chose another path, very close, slightly moved it, then you can fill everything in and the maps all commute, so you get the same map. In general when you do a more general thing you get an error from the curvature contrubutions. We'll get to that some other time. This we'll call flat because if you move things slightly there's no change to the holonomy.

So you see, now I'm sort of outlining the proof of the exercise. I would still like you to submit your own version. Take your point with the equivalence classes of paths, and a nice neighborhood. Then you have all the fibers over all the points in the neighborhood. Between any two fibers there's a canonical equivalence, which sets up a bijection of a fiber over x_0 and the neighborhood U with the union over $x \in U$ of the fibers over x. This is called a local trivialization, and is completely canonical. There's a natural topology, taking the discrete topology on F_{x_0} and the normal topology on U. So we can transport this over to $\cup_U F_x$. Then the claim is that all the paths landing in U is an open set in the compact open topology on paths, and the induced topology on the equivalence classes is homeomorphic to this topology via this canonical bijection. I said it more difficultly. Now I've given you the bijection. You just have to check that it's continuous in each direction. Another corollary to this picture is that, I'm calling this a covering space now. The definition is a map so that the preimage over a small enough neighborhood is a discrete set cross the the neighborhood. The fiber over the basepoint are homotopy classes of loops modulo H, which is the coset space $\pi_1(X, *)/H$. There is a canonical coset here which is H itself, which gives us a basepoint in the covering space. Call the cover $\tilde{X}$ and the basepoint $\tilde{*}$. So the corollary for path lifting is that this map $\pi_1(\tilde{X}, \tilde{*} \to \pi_1(X, *)$ is injective and the image is, well, call it H. If you look at the image and then go through the construction with H as the subgroup, there will be a basepoint respecting homeomorphism to this space.

What is equivalence of covering spaces? Suppose you have two $\tilde{X}$ and $\tilde{\tilde{X}}$. It's a homeomorphism, but you can ask for it to preserve basepoints or not to.

For any covering space the π_1 map is injective. You know when you project it downstairs, you can fill it in. You make it out of a bunch of very small movements. If you move this in, little by little, it will lift to paths to the same basepoint, and so you can keep lifting everything, and when you get it real small it's inside one of the neighborhood and fills in as you expect it to. So it's injective, that's the path-lifting property.

For a connected covering space, define H that's associated to it to be those closed curves at $*$ which, when lifted, lift to loops at the lift of $*$. This is what you might call the kernel of the holonomy map. Now, suppose you had chosen a different basepoint, suppose you worked on a different sheet. You could also choose the H for that sheet. There's some path between these two basepoints, this would go down to something like this, if one is a closed lift, then so is the composition with the path and its opposite on either side. So changing a basepoint changes the subgroup by conjugating.

Connected coverings with basepoint are in one to one correspondence with subgroups of π_1 and coverings without basepoints are in one to one correspondence with conjugacy classes of subgroups of π_1.

An example is that $\pi_1(\tilde{X}, \tilde{*}) = \{e\}$ corresponds to $H = \{e\}$. This is called the universal covering space. There's kind of a lattice of covers. Every subgroup corresponds to a covering space. There's a map as the subgroups get bigger and so there are maps between the covering spaces. The fiber over the basepoint is acted on by the group. The trivial subgroup gives you that two paths are equivalent if and only if the paths are homotopic. I should have said all that.

So what if the subgroup is normal? A subgroup is normal if, you take a covering space corresponding to it, no matter where you start, you get precisely that subgroup. No matter where you lift them, they lift to closed paths. This isn't true if you choose something that is not normal.

This construction has a lot more juice in it. Bundles with these holonomy operations. Okay.

Let's, I don't mind this abstract stuff, but maybe you guys are tired of it, let's do some examples.

When I was doing an undergraduate topology course, how do you prove that the fundamental group of the circle is $\mathbb{Z}$? How do you prove that the identity map is not nullhomotopic?

The universal cover is $\mathbb{R}$. So there's the universal covering space, the real line coiled around. The deck translations are just moving up and down, so it's $\mathbb{Z}$. The identity map, when you lift it to the covering space, comes back. If it were homotopic, it would come back. You still have to prove that this is a covering space, but then you're done. It's nontrivial that the line covers the circle. Riemann figured that out, it wasn't Joe Blow.

I was having difficulty showing that the closed upper half-plane was not homeomorphic to the open upper half-plane. If you remove an interior point, you can get a nontrivial covering space. If you remove a point on the boundary, you can't.

The punctured plane is still basically the circle. You can crush and move out to the circle. The universal covering space is the plane. The deck transformations are vertical translations by integer values. This is then the universal covering space. They had this example in the beginning of the nineteenth century. The logarithm is multivalued, but it's single-valued on the cover. Another example would be X the torus and then you add another translation in a transversal direction. Now every point can be reached by doing horizontal or vertical translations from a unit square. Then you have to identify the edges and you get a torus. This is not generic, this is sort of special. Let's do a generic construction. Take the torus, and you want to cut the torus, you can think with the circle that you cut it, and then lift that cut point and attach it to the other side of another copy of it.

You cut your space to become contractible and then attach the contractible pieces. If you cut along a longitude and a meridian you get the answer you want, but you might instead come back somewhere different than where you set out. So you get a hexagon. When you cut it open you get a six-sided figure, and now you have to fit them together. You can take a basepoint and connect every point in the hexagon to the basepoint, and lift the paths. Then we get a map of the hexagon into the universal covering space. The opposite sides that were identified correspond to nontrivial paths so the two sides won't be identified in the cover. You can continue lifting, and you get a paving of the universal cover by hexagons. This is a more generic picture of the torus. This is what bees discovered. That was supposed to be a joke.

Consider the compact surface of genus two. The trick was to take the two obvious ways to cut and then seperate them to make things generic. We're getting to a metric in a minute. The usual way of cutting the surface of genus two gives you twelve sides. But generically, you get an 18-gon. so we take a lot of those and start fitting them together in a trivalent fashion. When you do that once, you cover up two edges of each, and now you get sixteen on each ane of these. Do you think this looks like the plane? Thurston, this famous mathematician, wasn't that great in graduate school. He was a little weird and not getting any sleep because he had a kid. He was taking his first exam. They asked him if he knew what the universal covering space was. He drew the surface of genus two. He drew the analogue of this square decomposition. He drew eight sides. It still doesn't tile the standard plane. You have eight sectors coming in, so you need eight coming in on a vertex. He drew eight coming together very sloppily and they didn't think he knew what was going on. Really they didn't know what a universal cover was, not him, but he couldn't present it well enough.

So what about the surface of genus three? Make all of the vertices trivalent. You get four, five, four, and two, doubled. So when you cut open you get thirty. In general it's $12g - 6$.

The interior angle gets bigger as you get more sides. Basically when you're gluing three of them together, you're trying to glue three half-planes together at a point. But the angles are essentially straight. So it's hyperbolic. If you take the armpit of your shirt, that's two pieces, the total angle is 3π. So to make a T-shirt you put together three pieces of material like that. Before you do that, you get a T-shirt with a slightly bigger angle. All of these constructions look locally like a T-shirt, like gluing in space to get something two-dimensional with too much angle. You can fit things together but you get these funny corners. So what does that look like? It looks like the hyperbolic plane in a certain metric. We'll get to that next time.