Dennis Seminar
September 12, 2006

Gabriel C. Drummond-Cole

September 12, 2006

I'm going to start kind of inspired by some of this publicity about Perelman, discussing covering spaces and the fundamental group, and study how they control two-dimensional manifolds, and how in dimension three because of the Poincaré-Thurston geometrization conjecture, proven by Thurston, Hamilton, and Perelman, also controls in dimension three. If you try to study higher dimensional spaces, two dimensional complex algebraic varieties, you need more than the fundamental group, and you start introducing more tools of algebraic topology.

I skipped a step, but that's the historical development at the beginning of algebraic topology a hundred years ago. It continued. In the twentieth century, the methodology of algebraic topology, I'm going to draw a picture. You might have a lot of things, draw them as points, so there might be equivalences, which you draw as a path. Sometimes the paths or equivalences are themselves equivalent. If you keep going this way, that's the basic picture of digging into a space and getting the algebra out of, but also in algebra derived from geometric discussions.

Let's start. Now many of you here have heard of the fundamental group, but I'm going to emphasize the basic instruction in this first part because it has a lot of other applications. The idea of algebraic topology is to attach algebra to some spaces. I have a list of spaces. Typical spaces we want to start thinking about are, well, nonlinear spaces:

- Riemann surfaces
- Manifolds
- Solutions to algebraic equations in $\mathbb{C}^n$
- Algebraic varieties
- Mapping spaces, like the space of all continuous maps of the circle into a different space. This is an infinite dimensional space, but it's nonlinear.

That's kind of a good list of spaces. At first blush algebraic topology doesn't apply to linear spaces.

The first piece of algebra is that if you have two paths in a space, then you can sometimes combine them. You still get a map from 0 to 1 by splitting the path in half and speeding up each half to double speed. This partial composition, simple as it is, is pretty profound.

I want to do now the construction of covering spaces using paths. Just to keep those people interested, who, if you modify this construction slightly you can get all bundles with a, well, connection and a holonomy construction. So this has more juice in it than I'm using now, and that will come up later in the year.

So if X is some topological space, and we have to assume that it's locally path-connected and locally simply connected. This means every point has a neighborhood so that a path can be drawn between two points in the neighborhood and that in every neighborhood there is a smaller neighborhood where two any two paths between two points can be "filled in." You can drop this assumption but then some of the results won't be true.

Choose a basepoint $* \in X$, and this is very important. We're going to construct all the covering spaces of X and in particular the universal covering space. If I use my notes I can't think. Now consider, let R be an equivalence relation on all paths from the base point to some point $x \in x$ defined by setting two such paths γ_1, γ_2 equivalent if going along γ_1 and then backwards along γ_2, which we denote with a bar: $\bar{\gamma}_2$, then this falls in some given subset of closed paths starting and ending at the basepoint. This should be closed under inverse, so we want it to be invariant under $\gamma \mapsto \bar{\gamma}$. We want it to be transitive, so that $a,\ b \in S$ implies $a * b \in S$. Our big assumption to connect to covering spaces is that S contains all nullhomotopic loops. Then if you have a closed loop, there would be a map of the disk, well, a map of the square which sends $[0, 1] \times \{0, 1\}$ to the basepoint. If this extends over the disk, then we say it's nullhomotopic.

We've made all these assumptions, now to make the construction. The advanced remark related to this is that Milnor and some physicists like to describe theories on spaces of maps like this. They use a less brutal equivalence, that if you leave along a path and return along the same path, a nullhomotopy of zero area, you get into the holonomy discussion. Milnor has a nice paper at the end of the fifties. He showed that if you mod out by one of these relations, and also keep only a finite number of these excursions, you end up with a topological group.

Remember R was this equivalence relation. Let F_x be the set of equivalence relations. Take all the different equivalence classes of paths between the basepoint and x, and you put over x all of these equivalence classes. If you have another path from $x \to x'$, you can send equivalence classes of paths F_x to $F_{x'}$. Now if you have any path between two such points, you can compose and get a path to the next one. There is a map from the fiber of one point to the fiber of another. This is the result of choosing short paths. As you move the other point along the path, it moves around and finds another version of itself. Because of the way we have a big equivalence relation, if you move the path by a hometopy, then this won't change. So if you take one of these neighborhoods which come from the assumption, if you take another point here, then there's a unique path, shortest, so that there is a canonical equality.

Exercise 1 *There is a natural topolagy on $\cup_x F_x z$ and is homoemorphic to $U_x F_x$.*

Two paths are close if their endpoints or, I need to put a topology on functions.

[The compact open topology comes from taking as a basis the set of maps from compact subsets to open subsets of X.]

My mind is blank now. If the space is defined by a metric, then two paths are close if they are close metrically. Does anyone know how to define this correctly?

The advanced version for people too advanced to do this one is to try to discuss bundles with connection. If you don't know what this is, that's okay. Replace this nullhomotopic condition with the zero area nullhomotopic condition.

All right, now the example, the main example for today, I hope we get there, take S to be just the nullhomotopic paths. Two paths are equivalent if going around one and then back around the other, that's nullhomotopic. So the $\tilde{X}$ for this is the so-called universal covering space. And I should say that if, also, the fundamental group $\pi_1 X$ at the basepoint $*$ is all closed paths with composition modulo nullhomotopic paths. I should have added that over here, in the general case, the $\pi_1(X)$ at the basepoint acts on the $\tilde{X}$. It acts on every covering space, because if you have a basepoint, and then a point above x, then you go along a given path, and then around the path you were given, to get to another point over x.

In a general cover, well, there's action on this. All these fibers can be thought of as homogeneous spaces of this group, meaning that it acts transitively, but it doesn't make the fiber into a group.

I'm going to draw a group now, this is a group. You get cosets which give you partitions, like a foliation, and you can take the set of equivalence classes. The group acts, say, on the right for the left cosets. They are sets, not groups. Subcovers are in correspondence with subgroups of the fundamental group. If you remember this path construction you'll always be able to recreate this theory for yourself.

The kernel is sometimes called the isotropy group. We don't need all these words, it's just the construction.

Now this picture here, one fantastic property of this construction is that if X has some nice property, say it was a manifold with some kind of structure, then $\tilde{X}$ picks up any local properties that X has.

Let's recall that, if you've heard it already, that

1. a d-manifold is a Haussdorff space with a countable covering by open sets W_α each of which is homeomorphic to an open set $U_\alpha \subset \mathbb{R}^d$. There are interesting examples of non-Hausdorff manifolds, and then there are also examples of things that don't really come up in math, like the long line, or its product with itself. It's unknown whether that product has a real analytic structure, I think. The homeomorphisms $W_\alpha \to U_\alpha$ are called charts, and then the second thing to recall is

2. that given α, β, the overlap homeomorphism associated to this cover W_α,i s $\varphi_{\alpha\beta}$ is

defined between the images of $W_\alpha \cap W_\beta$ into $U_\alpha,\ U_\beta$ respectively.

3. X is an S-manifold for some structure on $\mathbb{R}^d$, if the $\varphi_{\alpha\beta}$ preserve the structure. For example, S could be differentiable.

Being a topological manifold is unique, there is no additional structure. But being a differentiable manifold you have to choose your charts carefully. There are many different structures on any space, say, on the real line. They may be equivalent but there are many of them. That's another advanced structure.

We could have differentiable in the holomorphic category, and then S would be holomorphic invertible with no zero derivatives.

We could also have, you could do symplectic, say that you have symplectic maps, ones that preserve a standard symplectic structure. You can modify this structure on $\amalg_\alpha \cup_\alpha$ and ask that it be compatible over isomorphism. That lets you get Riemannian metrics, we don't need to worry about that.

Then the main thing is that any covering space of an S-manifold is an S-manifold. For consistency I need to know the layers over a point are countable, but I can prove that.

So the covering space of a smooth manifold is a smooth manifold, similarly for a Riemannian structure.

Now let's define a Riemann surface. It's a surface, a two-dimensional manifold whose overlap maps in some chart are of the form $z \to f(z)$. Every place you have a $\phi_{\alpha\beta}$ it looks like this. Another logical thing is to take a maximal set in the set of charts, you throw in all charts that are compatible.

If you didn't make the countability assumption but you made the Hausdorff assumption, they proved that these are still paracompact.

Then there's this celebrated theorem about Riemann surfaces, that there are only three Riemann surfaces that are connected and simply connected, meaning that the fundamental group is trivial. They are $\mathbb{C}$, that's one, $\mathbb{H} \subset \mathbb{C}$, everything above a line, not including the line, and $\mathbb{C} \cup \infty = S^2$. Poincaré couldn't quite prove this, Koebe had to help him. And then this has roots in an older thing, Gauss showed that if you have a little piece of a Riemannian surface, there is a covering, by "Gauss' isothermal coordinates," where the maps are conformal but not distance-preserving. You can make the maps orientable, and then the overlap homeomorphisms will be orientation preserving and conformal. So the matrices are given by a single complex number. So any differentiable surface is a Riemann surface. Then you can take its universal cover, I didn't prove this but it's simply connected, so it's one of these three things. Now then you get a corollary, which is, well, we can do some examples, $\mathbb{C} - pt$, and then $z \mapsto e^z : \mathbb{C} \to \mathbb{C} - pt$ is the universal cover, and this is called the cylinder. This is also called $\mathbb{C}^*$. What are the deck transformations? They are $z \mapsto z + 2\pi i$, so this is the deck transformation, and you can divide by this, and also, keeping this one, divide by $z \mapsto z + \lambda$, any complex number not purely imaginary, then $\mathbb{C}$ modulo these two things, well, $\mathbb{C}$ modulo the first series of translations is the cylinder. If you mod out by the second set of translations, they'll be independent, and then this will be the torus. You can vary the λ parameter, so you get a number of tori. You also get $\mathbb{C} \cup \infty$, which is S^2. These we call the elementary Riemann surfaces. Putting this together, we get the theorem, there is a one to one correspondence between nonelementary Riemann surfaces and discrete subgroups without torsion of $PSl(2, \mathbb{R})$, matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant one, maps $z \mapsto \frac{az+b}{cz+d}$ with real coefficients. These are holomorphic translations of the upper half-plane, because the elementary ones exhaust the surfaces covered by S^2 and $\mathbb{C}$. There must be a subgroup, without torsion for transitivity and discrete so that the space below can be evenly covered.