Dennis Seminar
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You can start somewhere and consider the elements that take a particular chamber to the next one over. Those are elements of the fundamental group, and those are going to be the generators. Then the relations are obtained by, if you take, this is a codimension one wall, you can, let's pretend this is a manifold, you can push through this wall and then that one, keep pushing until you go around this edge here, until you come back into this room. That's a certain composition in this fundamental group. It takes this room back to itself so it has to be the identity. This has the property that anything preserving a chamber must be an identity. So the relations come from the corners. That reminds me, you know what one wall said to the other? Meet me at the corner.

It's an edge in dimension three but a surface in dimension four. It's codimension two. These give the relations in general, with the codimension one faces giving the generators. You can see that these are generators by moving through walls. If you're in a chamber, the element is obtained by transforming back, pushing forward, and then going back. So it's not a generator but by induction it's a word in the generators.

All the wall moves for the adjacent ones are made out of words, and so the next set of wall moves are in the group, and then the next set, you go back and then, uh, well, go here with these two wall moves, but that's in your generators. So that proves that these are generators. Then it's obvious that going around a corner gives a relation. Then there was this argument that any two paths between two chambers can be deformed one to the other, only crossing through walls, maybe you have two paths, one through Port Jefferson and one through Smithtown, you can move back by crossing corners and never corners of corners. When you pass the corner you sort of use that relation. So this way you see we have a presentation. The corners are conjugate so you can move the corners back to the initial corner. You use the transported letters to do this so these relations can be a little complicated. It would be good to imagine how to write out a rigorous proof. A deformation of paths doesn't have to go through codimension three.

Any questions about that?

Andrew suggests that you can think downstairs, projecting points downstairs to a finite set.

In dimension three, removing those doesn't affect, every path is uniquely homotopic to a path missing those points. Homotopies can be deformed slightly to miss those points. In a four-manifold you can remove a circle or a graph and not affect the fundamental group, in a five manifold you can removie a surface.

I'm a little worried that some of you don't understand what I'm saying well enough to ask questions.

[Last time we were trying to [unintelligible]]

In any dimension, we have a dual chambering to a given one. I have to make a lot of assumptions. The nicest one is to take the hexagonal tiling. I get too much fun out of drawing this. The dual replaces hexagons with vertices and then edges with edges crossing them. Then it has a face for each vertex. You can imagine doing that in general.

[What about boundary?]

There are two kinds of boundary conditions, roughly, it's like, [unintelligible]

In three dimensions, I don't want to draw the cubical one, but it's self-dual.

[Picture of the three dimensional case.]

In dimension three we get two presentations, one for each chambering. Obviously the number of generators in one case is the same as the number of relations in the other. This implies some abstract property which I don't remember about the fundamental group of a closed three-manifold. The $\sup\{g-r\} \geq 0$, so there exists a presentation with more generators than relations.

So we can do $\mathbb{Z}$ and $\mathbb{Z}^3$. What about $\mathbb{Z}^2$?. One knows the answer is no. What about $\mathbb{Z}/n$? There are up to $\varphi(n)$, from lens spaces. All one-generator groups occur. What about $\mathbb{Z}_2 \oplus \mathbb{Z}_2$? The answer is no. Similarly, any $\mathbb{Z}_n \oplus \mathbb{Z}_m$, if m and n are not relatively prime, will require two generators and then additional relations. So this almost exhausts the Abelian groups. This is a reasonable exercise, I think.

Exercise 1 *Find all Abelian groups that have this property.*

This is a good joke for mathematicians to write down, $2, 3, 5, 7, 11, 13, \ldots, 41, 43, 47, 53, 59$, and what is the next element? It's not 61, it's 60. That's a mathematician's joke. This is a rather, one has a list of all the three-manifolds with finite fundamental group, these that I've mentioned and the three platonic solid groups. Part of the wonderful theorem of Perelman is that these are all of them. It's not obvious why these manifolds, why A_5 extended by $\mathbb{Z}_2$ has this presentation.