Dennis Seminar
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Okay, so I'll return to the free product and amalgamations next week. I wanted to talk about something a little different today and then more on this on Thursday. This is about a bunch of attempts to prove the Poincaré conjecture. Some of these questions were harder and some more general than the Poincaré conjecture. So then many of these conjectures are now solved. Andrew's going to give a presentation on Thursday about this.

So let X be a 2-complex, so it has vertices, edges, and faces glued on with attaching maps. Now if you, an example we talked about is if we have a presentation of a group, they used the term balanced presentation, then you get one of these two-complexes that has one vertex. Now, this could be a presentation of the trivial group. So suppose that this presentation P describes the trivial group. Then $\chi = 1 - \#g + \#r = 1$ in the balanced case when r and g have the same number. Then by algebraic topology, using secret calculations, this impies that X is contractible to a point. What this means, a space X is contractible to a point if, you can choose a point, and bring all the points in the space continuously into that point. There's a map $X \times I \to X$ so that $X \times \{0\} \to X$ is the identity and $X \times \{1\} \to X$ is a constant map. So then you can pinch $X \times \{1\}$ to a point and so this is the same as saying you can map the cone on X into the space with X mapping via the identity. I'm not expecting you to understand why this thing is contractible, just take my word on it that it is. There's a much more geometric way to see that a space is contractible. If I take a space that is a bunch of pages sticking out of a spine, I can collapse to the spine and then collapse the spine to a point. If you have one exposed face of a cell, you put your fist in it and collapse it. So if you have a space which can be collapsed with a sequence like that, it's certainly contractible. One can ask to what extent a cell complex is collapsible if it can be reduced to a point by elementary collapses. Now, properly speaking, this course is advertised as intermediate algebraic topology, and this is geometric. This is not known. The Poincaré conjecture is equivalent to saying that removing a ball from something simply connected you get something collapsible. It will be contractible. Let me just show you something that is not collapsible. Here's a two-complex which is contractible but not collapsible.

The example is called the dunce hat. You take a triangle, and identify the sides. You get an ice cream cone, you can glue the point to the edge and get a sleeping hat, and then you glue the two circles together. The exercise is, construct a contraction of the dunce hat.

Here's another one, the house with two rooms. This is contractible, clearly, but it has no exposed edge so it's not collapsible. There's a theorem that a manifold with boundary that is collapsible is equivalent to the ball. So people thought, maybe you can understand these things. You can kind of uncollapse these things. Some picture like this might be a counterexample to the Poincaré conjecture.

[Does thinking about these things lead to a proof in higher dimensions?]

That's kind of what Stallings did. Here was the conjecture. So a contractible, it's called AC for some reason. A two-complex, a contractible 2-complex which can be embedded in a three-manifold is 3-collapsible. This means that you have something, so I'm going to allow myself to add to this space 3 dimensional pieces, and then it would be collapsible.

Then a theorem of Wall and Whitehead is that if an n-complex, a contractible n-complex is $n+1$-collapsible for $n \neq 2$. Then the fact AC is equivalent to the Poincaré conjecture. So now in the theorem you can do it in the case $n = 2$ just for the case that the complex is embeddable.

It's not to be ruled out that there is a more intelligent natural proof.

Anyway, we have this interesting fact that something that satisfies the local properties of being embeddable, then it's collapsible. I gave this criterion that was sort of semi-local. I gave this criterion which was about planar graphs. Things look like a surface or three pages coming together, maybe with another transzilsal page. You have four points along the singular directions, and every pair can be connected without passing through the singular lines. Any of these things can be locally embedded in a manifold. You do this at every singular point, and now you have to worry about, you have three pages coming together. I'm using the fact that there are only two cyclic orders of three things. These fit with either a rotation or a flip. The surface parts can be thickened with an interval.

There are people who study standard complexes with standard singularities. Let me mention a different thing now. The point was to introduce the notion of contractible. Algebraic topology is exactly set up to answer the question of whether two spaces are contractible. Collapsibility is subtler, although, this theorem I said here for $n \neq 2$ uses algebraic topology. For $n = 1$ it's trivial, it's the same in dimension one. For a two complex it's still not known.

I realize I've skipped a layer of structure called simple homotopy theory.

These were attempts to prove the Poincaré conjecture. You have this two complex, it has the right Euler characteristic, you want to show the thickening is collapsible. There's a bunch of conjectures Andrew is going to talk about. The idea is that when you have more room you can avoid these impediments. The four dimensional one is still not known, it's known topologically, but you want to know in terms of cells or the differential structure. Is it smoothly the sphere? That's not known in dimension four. It's known in every dimension that if you remove a ball from the sphere, that you can collapse it down to a point. You can do this continuously in dimension four but not necessarily with cellular pushes.

The theorem Thomas mentions, Freedman, uses infinite processes, where it's really incredibly complicated. It's nowhere near anything that has any finite structure. There are little geometric structures in it, but they involve surfaces on surfaces on surfaces et cetera. This is a very bad homeomorphism. The genus of the surface goes up in an incredible way. This grows in a noncomputable way. The proof doesn't even indicate at all that there should be a well behaved version. Four is terra incognita. There is no upper bound for how bad things can be.

Everything behaves the way it's supposed to in dimension three, but the proof is five hundred pages or whatever. The simple ideas I've been discussing get you into the problem right away. Balanced presentations of the trivial group, do they have the property that when embeddable in a three manifold they are three-collapsible?

Anyway, Stallings, he was a professor when I was a graduate student, he's still around, so his proof strategy involved reducing Poincaré to group theory problems. In a paper, "How not to prove the Poincaré conjecture," Stallings wrote about not being able to judge his own proof. Whenever I prove the Poincaré conjecture I go out and take a break or whatever, have some coffee. I know in the light of day it will be wrong.

So in 1998 working with Moira Chas we worked out, well, if you take a surface, even one of genus two, we wrote down what its fundamental group is, it has four generators and one relation, it's as big as the free group on three generators. It has lots of conjugacy classes. There are obviously lots of structure to these things. They can even be simple curves, but very complicated. You can imagine, these are very complicated. They will be very long, but if you take a homeomorphism, every simple curve is homeomorphic to one or another of two simple curves, according to whether the curve is separating or not. They can look complicated but if you move by a homeomorphism, you can get them on a single torus connect sum component. Thurston developed a sort of dynamics of these. Here was Stallings' question. Let π_g be the fundamental group of the genus g surface. Picture this embedded in space. There are two natural maps to the free group on g generators. You can glue the boundary to its interior, in which case you get fundamental group free on g generators. It turns out that the complement of the solid thing which gives you another homomorphism. Then you can form the Cartesian product homomorphism into $F_g \times F_g$. Because the three-sphere is simply connected, this map is onto. If you take two of these, this map is onto. The ambient manifold is simply connected if and only if the map is onto.

The two conjectures are, first, every surjection has a nontrivial embedded curve in the kernel. Second, and this has been known, this is equivalent to the Poincaré conjecture. There's now a stronger conjecture, that there is only one such homomorphim, up to an automorphism of the domain.

Okay.