Dennis Seminar
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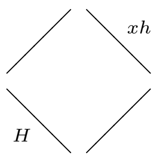
I've bypassed the more elementary results of Van Kampen's theorem. I'm interpreting this proof at some intermediate level, telling us more about the structure of the group. The hypothesis of injectivity gives you some control over what's going on.

I want to state a theorem, but before that I want to state a definition.

Definition 1 *Let H be a subgroup of G. Then a complete transversal is a choice of one representative from each right coset of H in G. I'll say, nontrivial. There's a slight imprecision because of the coset H itself. You have*



I want to choose the identity in H.

The theorem is that

Theorem 1 *if $X = A \cup_C B$ with everything connected, and $\pi_1(C,*) \to \pi_1(A,*), \pi_1(B,*)$ injections, then each element in $\pi_1(X,*)$ has a unique canonical form alternating s_i and t_i and then ending with h where $h \in H = \pi_1(A \cap B = C)$ and t_i transversals for $\pi_1(A)$ and $s_1, \ldots, s_2$ are transversals for $\pi_1(B)$.*

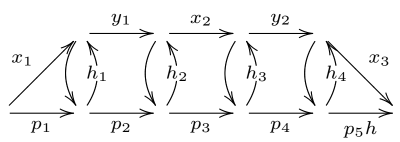
It's very easy to take words and manipulate them to get them into this form. The uniqueness is hard in the books, but it follows from the picture. We're proving Van Kampen's theorem and more.

This is doing one of the exercises. I thought that this was obvious. I'll explain why it's obvious and then you can write it up as homework.

Theorem 2 *The conjugacy classes have a canonical form of a ring of alternating transversals.*

Maybe that's false.

Let's concentrate on the first theorem. Let me write a picture, I'll say what it means as I go along. I'll read from left to right. So, uh, suppose I have a string of words on alternating sides of the wall. Now, well, the first x_1 is given by $p_1 h$ for some h. I'll let p_i stand for either s_i or t_i, whichever it should be. So x_i is in a group, so it's in a right coset, so I can write it as a transversal times an element h_1. Now I look at $h_1 y_1$. This is a group element, so it's in some coset, so I can write it as $p_2 h_2$. So I have $p_2 h_2 = p_1 y_1$. Now we do the same thing again. We can write this as a transversal to h_3. Then I can put in p_3, and so on.



So this shows two things, actually. Suppose I hadn't put the labels h_i on there. It shows any element can be written in this form.

So that shows that each element has a canonical form.

This is in a way, this algebra tells you how to homotope your guy into canonical form. You only have to keep the endpoints fixed. So you can take x_1, you can take the endpoint, and move it around h_1^{-1}. This tells you how to make a homotopy in the wall. That will move to p_1, and the endpoint of y_1 gets dragged around to be $h_1 y_1$. We move its endpoint around h_2 backwards to move us to p_2, and continue in this way. We're performing these homotopies of the wall to move us over to this form. We could think of them as homotopies on their respective sides. Now the punchline is, if we have any homotopy, our geometric lemma says that after doing the disk exchange property, every homotopy has this form. If you took some homotopy to another set of ps, you could put it in this form. So you'd have to satisfy the same properties and this would be unique. Say it again. Suppose you took a homotopy to $p_1' p_2' \dots$ Everything has to be the same once you put that in this form.

This theorem has a name. The existence is trivial. The uniqueness is, I don't know how they do it. They may have this picture in mind, I can't read that book, Moira can read it, I can't.

There is this group $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. This goroup is $Sl(2, \mathbb{Z})$. So now you have two cosets in $\mathbb{Z}_4$ and $\mathbb{Z}_6$ has three. So you have only one choice on one side, and two choices for the other side. You can factor this in various ways, you might want to see what the factoring is. Then $PSl(2, \mathbb{Z})$ is $\mathbb{Z}_2 * \mathbb{Z}_3$. When the subgroup is the identity, then these elments are just the elements of the group.

Now you have to help me, let's do the conjugacy classes. You have to wear a mask next time so I don't know what you're thinking.

I like free homotopy classes. We'll deform every conjugacy class to go through the basepoint. You can factor into group elements. There we need H to be connected. Now I just start somewhere. Let's imagine, let's start somewhere. I could apply that procedure that I did before. I have $p_1, p_2,\ h_1, h_2$, as before. I get all of these things. Here I had e before and now I have h_6.

I'm going to write down statements I remember. Free homotopy classes, i.e., conjugacy classes in $\pi_1 A \cup_H B$ consist of the conjugacy classes of H modulo equivalence on either side. So π_1 is injective but the conjugacy classes need not inject. That's part one. Part two is, cyclic words in transversals plus one H factor modulo cyclic permutation and H-conjugacy.

I'm sorry, we already have the proof, we don't have the statement. This is simultaneous H-conjugation. Let me point out how that is a much more subtle issue than normal conjugation.

Take S_n. What are its conjugacy classes. These can be written as a disjoint union of cycles. By relabelling you can see a permutation. Suppose I give myself two permutations. If I ask what the conjugacy class of a pair is, there are many more invariants of that that we don't even know.

Consider the group generated by the two elements p_1, p_2, you can attach to that any finite group with two generators. So now we have to classify every finite group with two generators. That's hard. When you do simultaneous conjugation, there are a lot of equivalences.

I think I'll stop, start on the next topic. Any questions?

This is not part of the class, I'm just going to talk some more.

This question is related to thinking of a finite set of circles lying like strings on the floor. You go and pick them up, cut them, and then reconnect them according to some permutation rule. You now have another bunch of circles. All the cuts, there's one permutation, which is, well, label the cuts up to n. The cuts live on the circle. These circles are directed, so now you get directed circles. You also have another permutation indexed by the indices of what cuts go where.

On the other hand, I claim that all of these things can be viewed as a surface you can build, which has one critical level. Its also equivalent to giving a surface mapping to I by mapping, and if you filled in the surface you would get this. It's also equivalent to drawing one of these cell decompositions of a surface, where all the valences are even and it's two-colorable. You can color the upper ones black and the lower ones white, say. It's the set of two-colorable cell decompositions on a surface. Grothiendieck has a discussion with combinatorial objects related to Galois theory. That's an intermediate level discussion of the symmetric group. There are people who talk about simultaneous conjugation of strings of matrixes. So there is something to say about thi.s. If you have an arbitrary permutation, you have $n!$, but if you're just talking about conjugacy classes, you have partitions of n. Those arise everywhere. Let's stop.