# Dennis Seminar <br> October 17, 2006 

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October 20, 2006

Take a space with a wall in it, $M$, seperating the space into two parts $X$ and $Y$, and we assume that the inclusion of $M$ into the sides induces an injection on $\pi_{1}(W)$ into $\pi_{1}$ of $X$ and $Y$. I want to discuss maps of an annulus or a cylinder into the same thing.

So $X \cup Y-W$ should have two components, the closures should intersect along $W$, that's the other condition.

There are four cases for this annulus. I'm going to consider that there's a basepoint in the boundary of the annulus, on each component. I want to be able to break the boundary up into pieces mapping into $X$ and $Y$. Then I suppose that the maps of the circles can be filled in continuously with maps of the annulus into the whole space. I have not used the assumption about injectivity. In one case I'll want to consider, I was going to state that later, in one case the base point stays at the base point and in the other case it doesn't. Then there are, well, there are actually four cases. The other question, besides whether the basepoint stays fixed, is whether I have one in $X$ and one in $Y$, or a bunch of back and forth.

The statement is that we can deform the map of the annulus or cylinder, relative to the boundary, so that it looks like cylinders stuck together, each of which is contained entirely in $X$ or entirely in $Y$.

In case $A$ when the homotopy is between two taut guys, if you had an annulus, you can divide it into regions like spokes, which are colored alternatingly.

Recall that you had nice intersection with the wall, and you moved this to get transversality along the annulus, so it should cut up a bunch of curves. By transversality this will be a one-manifold.

Let me comment on case $A$, anything that travels over on one side, goes and comes back, can be homotoped back into the wall. So I was supposing I didn't have those. So everything goes across. They can't peter out in the middle, either. Without doing anything it shows you have the same number. The other circles inside, we showed how you eliminate them. The Jordan curve theorem, which is nontrivial, is needed here. So then the innermost disk is a curve on the wall and you have a map of the disk in. Then you use the injective hypothesis,
changing the basepoint, and it bounds a disk on one side so it bounds a disk on the wall. So we can redefine the map to take this into the wall. You pop the map over the hole. So the statement is wrong. We have to be able to pop, not just deform. I can get rid of the holes to make the statement true. So then you can push the disk off the wall back into the other half and then the circle is gone.

In case $B$ we get, there, and then the cuts across cannot happen. Everything has to go across so there can't be any meridians in type $A$, in type $B$ everything is a meridian.

Let's go back to case $B$, I have the map of the annulus. If I suppose the bottom arc of the annulus was mapped into a given arc, in the first case you get that the element in $A$ is equivalent to something in $\pi_{1}$ of the wall. If two elements are homotopic in the whole space, one in each of the two pieces, then you can express this in the obvious way, as a homotopy across the wall. Since $\pi_{1}$ is injective, the two wall elements are homotopic.

Okay, so in the case $A$ you get these lines across which are not necessarily staying fixed at the basepoint. So we have loops in one side which are being homotoped, one to the other. If the wall were simply connected, you could squeeze the homotopy to a point in the wall. That would be the free product. Then the elements of the fundamental group would be words in the combination of the fundamental groups of the components.

If we ignore basepoints, then the annulus argument says that the map of the annulus can be changed to something freely homotopic to something in the wall. These things might not be conjugate in the wall, but they are conjugate in one side or the other.

Now this statement that Nate made, the formula $\pi_{1}(X) \cong \pi_{1}(A) *_{\pi_{1}(A \cap B)} \pi_{1}(B)$, that can be shown using less argument than I've made, but I've done some more, you get more information. If you have a group and a subgroup, you can partition the group into the cosets of the subgroup. So the group is the union of $H$ and $a_{1} H$ and $a_{2} H$ and so on. This is like dynamical systems. You have the group acting on the whole group. You have the orbits of the action of $H$ on $G$ by left multiplication. If two orbits intersect, they're equal. Each translation is, well, if you have a group acting on a space, they're equivalent if you can move from one to another by the action. So for example we can deduce a little bit, like that if one of the edges stays at the basepoint, as you cross the annulus, then the two words $x$ and $y$ are in the same (right) coset:


What about the other cases? I'm stuck myself. Time is up, thank God.

