Dennis Seminar
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Gabriel C. Drummond-Cole

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Take a surface, nogatively curved, then you look at what is called the geodesic flow, then this has an ergodic property. You go densely through the surface, but also in all directions, for almost all paths. That's if your surface is compact or even finite volume. It will go out the cusps and come back in. So the probability that it goes straight out the cusp is zero. If you measure the distance out the cusp at time t, you want to find a function $\varphi(t)$ so that $\limsup \frac{d(t)}{\varphi(t)} = 1$ and that function is $\log\ t$. If you change variables to the Euclidean you get something more like $\pi\sqrt{N \log\ \log\ N}$. The ideas is sort of like the idea of a tree. The hyperbolic plane is something like a tree. Every time you get to a tree, you go on either zero or one. Folding this down onto a compact manifold, you get something similar.

When balls are bouncing off one another on a billiard table, you cut a hole in the middle and bounce off it, and unfold this. So negative curvature is created by balls bouncing off one another. Trying to find a geometric explanation for the thermalizing that occurs. That's explained by this idea.

So I wanted to mention an idea, we've been talking about using the fundamental group, and we have a list of all three-manifolds, and they're obtained from certain basic pieces.

So you want to start with Lie group actions with compact isotropy, and then you have at most six dimensions, three from rotations and three from translations. You have $SO(3)$, really we're talking about Lie algebras. In dimension two you have three possibilities, $SO(3)$, $\mathbb{R}^2 \ltimes SO(2)$, and $PSl(2, \mathbb{R})$.

There are other groups that have this properties, you only want to take the maximal ones. I think that gets down to the eight, but I'm willing to allow that there may be more.

So you have three of these things in dimension two, Locally, near the origin, $SO(3)$ and SU_2 are the same. I can get five examples out of these three things immediately. I can cross with $\mathbb{R}$ to get the lower level. In the curved cases, I get a crossing which is different from the other possibilities. You can get the unit tangent bundle as well. You'll still get something homogeneous. You can use the Levi-Civita connection to move tangent vectors along paths. You can do this in the unit tangent bundle of the spherical and hyperbolic space. There is this family of horizontal subspaces. You take a piece of it, and cross with something, topologically, but not metrically. It's not the same in all directions.

What is the unit tangent bundle of the 2-sphere? It's $SO(3)$. Take the double cover and get S^3. You can also do this over the hyperbolic space, and in both cases it will give you the twisted product with $\mathbb{R}$ which comes from the contact structure.

So these are:

$$\mathbb{H}^3(6)(Psl(2,\mathbb{C})$$

$\widetilde{Sl(2,\mathbb{R})}(3)$ Heisenberg(3) $S^3(6)$ with Lie group $SO(4)$

$\mathbb{H}^2 \times \mathbb{R}(4)$ with $Sl(2,\mathbb{R}) \times \mathbb{R}$ $\mathbb{R}^3(6)$with the Euclidean Group $S^2 \times \mathbb{R}(4)$ with $SO(3) \times \mathbb{R}$

$$Solv(3)$$

The Solv comes from taking a torus cross an interval and gluing its ends together via something nontrivial. There are three types, an invariant geodesic, an invariant point, and an invariant point at infinity. These go to Solv, Euc, and Heis, respectively.

I've used up the hour, maybe I should make a statement. When you have, there's a rough geometry way to look at things that distinguishes all these cases. I wanted to talk about rough geometry. So geometry means the objects are metric spaces. Suppose you have two metric spaces and want to talk about them being roughly equivalent. One says that there's a homeomorphism whose distance ratios are bounded above and below. This is called quasi-isometric. We want something even rougher so that if there are a discrete set of points in the plane, at least a certain distance apart, all over the plane, so that everything is within a certain distance from one of these spaces. So a rough isometry you'll have a relation between these two things. So there is a uniformly bounded set related to a given point. You have a rough map, an approximate map. The distance between two clumps and their corresponding clumps, those ratios should be uniformly bounded. So then you can go around, remember, we had pictures, there's another fact due to Milnor, there's a rough geometry of a finitely generated group. You can choose your generators, and then find a metric via the shortest length words. If you take another generating set, then you can rewrite the one set in terms of the other. It only gets multiplied by a fixed word length at most. So two metrics for the same group with different generators are roughly equivalent. This is an idea of Gromov. It started with Milnor. He showed that one invariant is that you can go out to distance N, and that is some function, it can have polynomial growth, exponential growth, and this is invariant of the generating set. He also showed that if you have a compact manifold and its universal cover, you have some look, some rough geometry. The rough geometry of the universal cover is the same as the rough geometry of π_1 because you have these compact fundamental domains. Now we could start thinking about the rough geometry of the group, and things fit together nicely.