Dennis Seminar
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We've been in the situation of denial in this class. We haven't discussed torsion. It's very convenient to have a more universal picture of the universal cover. We have this irredundant list of three-manifolds, we would like to generalize. Let's say we have torsion. You can make a right angled pentagon in non-Euclidean geometry, and make triangles with angles π/q. I gave a homework problem to prove that if you start reflecting in the sides of these, you get a tiling. So assume you have that.

I have, I'm in denial about orientation, so since I have these reflections, I have orientation reversing elements in the group. Take the subgroup of even words in the reflections. You could color reflections, black and white, and it takes an even number of times to go back by 2π. Let me not be in denial. I have the group generated by reflections and the subgroup of even words in reflections. We could just talk about both of these at once. What are the relations? You could walk your tile to another in two ways, that gives you a relation. So now, you can move, if you move a path, you slowly move a path across a vertex and you get a relation, all the vertex relations can be conjugated back to vertices in the original group, so you get $\{r_i\}$ which satisfy $r_i^2 = 1$. You also, looking at a vertex, you get rotations by the angle $2\pi/n$. Call the reflections r, r', then rr' is the rotation by $2\pi/n$. There's a wonderful thing, if you do reflections in disjoint lines, you get translation through twice the distance. In some complex world these would be the real and imaginary parts of some complex log.

There's some deeper algebraic about the reflections. You compose two and you get twice a quantity related to the two reflections. Anyway. In some sense there's a more general thing. You get a relation for each vertex $(r_i r_{i+1})^n = 1$. Those are the vertex relations. If you reflect around until you get back here, you get $2n$ reflections and you get back to the identity. If the angle were $\pi/3$, you would have this as a fundamental domain. Any two successive reflections are the rotation by that angle.

So, ahh, now if we took, these are all the relations. Now we can take the subgroup of even words, and we've described, that's described by this presentation. Then we can talk about the fundamental domain of the subgroup of even words. It would have to be symmetric about a line.

You have this triangle in non-Euclidean geometry precisely when, well, $(\frac{1}{p} + 1q + 1r - 1)2\pi$ is the area times the curvature. So $2, 3, 7$, well, $2, 3, 6$ is Euclidean, and then $2, 3, 5$ is spherical. The area of the sphere is 4π, and the sum is $\frac{31}{30}$, and it takes thirty of those to cover the sphere. I was expecting to get twenty or something. oh, you need to multiply by what to get 4π? So we need 120 of these to tile the sphere.

This is 12×10, you draw perpendiculars to all the sides and angles on the dodecahedron blown out to the sphere. This gives you $\pi/3$. There's a famous result that if you have a symmetry group of a genus g surface, it has order at most $84(g-1) = 42(2g-2)$. This is the smallest area of a fundamental domain.

Now the group is generated more by rotations once you take this larger fundamental domain. When you take the quotient of the hyperbolic plane by this group, you get, topologically, you get what you get by sewing two triangles together by their edges. You get a hyperbolic sphere except that it has three sharp corners. So you can kind of think that locally, well, this is not the two-sphere geometrically, even though it looks like it. When you have a group with torsion, you can still form the quotient but you might have corners in it. You would get boundary if you allowed reflection. Now this picture has been around for a long time, but Thurston figured out you can generalize the notion of path and form the universal cover. You don't know the picture above, and you just start making the paths. You have these amazing constructions to create the universal covering space, that you can, in the 1980s, i was in the room when this was named, people don't understand this, these objects were quotients of groups, but Thurston said that looking downstairs you can reconstruct the group and the space above. Here you need a topological space and a a coordinate system which is a Euclidean patch divided by a rotation group.

But there's not exactly a group attached to this point. For each point you have the subgroup that fixes that point, and all of them are isomorphic. To move from one to another you have to conjugate by an element, and there might be more than one such isomorphism. There is not a unique isomorphism. Intuitively you'd like to attach a group to the point, but you can't.

And now, see, now you want to start talking about paths. You have these two triangles that fit together. If your path goes through a bad point, what you have to do is assign which sector you're continuing in. So you give a group element to each bad point that the path goes through. Then you apply it to see which out going path to take.

This silly course on algebraic topology, all we've done was define the path-lifting property, homotopy classes of paths, and I'm just extending that to a larger class of spaces, called orbifolds. It's a little subtle, but the picture is sort of clear. I realized this didn't exactly fit with the discussion we've had, it doesn't have a well-defined group. If you look at the situation above, you don't exactly have it.

There's a cocktail party notion of orbifold floating around. What kind of equivalences do you have with the quotients? There's something called a stack in algebraic geometry, like categories, you have a bunch of things sitting over the point. It took someone of his level, how do you add to the theory of covering spaces?

Okay, there's a concept. Its name is orbifold. I'm not exactly comfortable with what definition I want to take. When the groups are Abelian, if you take it to be orientable, there really is a completely canonical group attached to a point.

We have Travis and Kate working on this, so they'll work it out. There is this concept of geometric objects with corners, and we want to include that. I was at a topology conference over the weekend, and I was checking my understanding of this with one of Thurston's students, Dave Gabai, he suggested, well, we're trying to make an irredundant list of three manifolds. There are these, what I called jewels, and then there was jewelry. There were these two stages. Now these things, there were actually eight varieties of jewels. I presented two varieties last time. There are actually eight geometries here that are relevant. This means the space locally have some geometry and all the points look the same, meaning they're isometric. So it turns out, in dimension two, we were playing around and we had, you can take the three from dimension two and get six of the eight. We can take the three and cross them with the reals. That creates a little piece of geometry that is clearly homogeneous. Those are three of them. Now there's a way to do a twisted product with $\mathbb{R}$. I don't know how to draw it, it's a twisted product. intuitively, as a differentiable manifold it's the same. But the first one was defined by taking the product of the metrics. You could instead choose a varying horizontal subspace, and then take the inner products induced from the projection, and declare these two pieces to be orthogonal. It turns out there is a field of two-planes. This field is called the canonical contact structure.

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This thing turns out to be totally homogeneous. The product structure could be filled in by planes, here you can't find surfaces tangent to these planes. It turns out that up to diffeomorphism there's only one way to do this. So do this over the spherical Euclidean and hyperbolic surfaces. We get the three-sphere, then a Lie group (the nilpotent group) and then you have like the metric on the unit circle bundle on $\mathbb{H}^2$, which is related to $SL_2(\mathbb{R})$.

These six geometries, there is, they are like filigree, that's a type of foliated gold jewelry. There is this family, and if you use the metric you get this twisted geometry, you can say it directly, you wouldn't see this as being the same idea applied to these three geometries. There are two more geometries that we don't get like this. You can get the hyperbolic non-Euclidean geometry, like a triply curved object, and then there's the other Lie group geometry, called Solv. In this you have a box, your box gets squeezed in one dimension and stretched in the other as you move along a perpendicular line.

Apparently this is important in computer science. Let me say what the manifolds are in this geometry. We've discussed the manifolds in $\mathbb{H}^3$. You get examples there. In the Solv case, you can, well, this is an irrationally filigreed surface. This is a torus over a circle where when you come back you go by a matrix like $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This has irrational eigendirections. On the torus you still have the same picture. There are two irrational foliations parallel to this. That takes a square and maps it to a thin thing. This is how you study, if you take any diffeomorphism close to this one, the structure is the same as what you started with. You get a three-manifold, go around, and glue it back by this diffeomorphism. This thing is foliated by these lines, so it's kind of filigree too, but in an irrational way. This looks like a T^2 bundle over a circle, this is a matrix with two real eigenvalues. A so-called hyperbolic transformation. There is a Lie group associated with this, which has a projection to $\mathbb{R}$ with kernel $\mathbb{R} \times \mathbb{R}$, and the single $\mathbb{R}$ acts on the other by this transformation. Over the reals you could just say, $t(x, y)$ is by this matrix. It doesn't have to be integers. It's $(\lambda x, \frac{1}{\lambda} y)$. There are discrete subgroups like $\mathbb{Z} \oplus \mathbb{Z} \to \Gamma \to \mathbb{Z}$, where the single $\mathbb{Z}$ acts by the integer matrix. This creates, what happens here, this group is exponentially big because if you take an integer like ten, you have to take the tenth power of the matrix, and so you get a big eigenvalue, something like the tenth fibonacci number. The unit tangent bundle over the hyperbolic plane and the hyperbolic one have exponential growth, the twisted Euclidean has polynomial growth, the S^3 is finite, well, out of time.