Dennis Seminar
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Here's the plan, this is going to explain the consequences of the Perelman affirmation of the Thurston geometrization conjecture and the Poincaré conjecture. Is there any homwework to hand in? Nice. Oh, and, here's a paper by Warren Smith on the shape of the universe, a physicist. He was trying to cut down on the number of possibilities to a finite number. Is it compact or not? Charge balance is accounted for by Gauss' theorem if the universe is compact, no boundary. But anyway, if anyone is interested in physics, well, this man is around here, he's kind of like a Stony Brook denizen. Anyway, you can borrow it from me.

Let's recall $2D$ non-Euclidean geometry. With few exceptions, any surface is the non-Euclidean plane divided by some discrete group. You could make tilings starting with a fundamental domain. If you start with a regular n-gon small, you can get, it's a lot like a Euclidean pentagon, $108°$ angles. You can draw geodesics out to the boundary, and connect the endpoints a certain distance out with geodesics, and you get a regular pentagon with different angles. This goes smoothly out to $0°$ so you can get $\pi/2, \pi/3, \pi/4$ angles, whatever. With a square you could get $\pi/3, \pi/4$, and so on. The triangle would give you $\pi/4, \pi/5$, and so on.

If you folded up a square with $60°$ angles, it would look like a torus except at one point, where four $60°$ angles come together, a cone point. This is called an orbifold. This is an efficient extension of manifolds, because examples are often orbifolds that must be unrolled to get manifolds.

Now, let's go back to this right angled pentagon, and do a different thing. Whenever you have angles in the boundary of something, you can start reflecting, and you can imagine keeping on reflecting through the sides. Two reflections gives you a rotation of twice the angle, just like two reflections like this give you a translation.

When you start going around a vertex, if you have $\pi/4$ you will come back by the identity map. Intuitively you believe that if it didn't fit when it came back around, you wouldn't get a tiling. I should probably prove that it gives a tiling.

Exercise 1 *This is due to Poincaré, and people debate whether he actually proved it. Show* *that a polygon with angles π/n generates a reflection group tiling. So here I mean of non-Euclidean or Euclidean geometry.*

So for example you could pick the square in the plane, and you would have four letters. There would be some commutation, of adjacent edges. If you have pairs, if they appear the same number of times, you get a translation, there's a subgroup of index four, which is $\mathbb{Z} \oplus \mathbb{Z}$, which is the normal reflection group of the torus. Some finite index subgroup is the normal group of deck transformations. On the line, you have two reflections, and this is $\mathbb{Z}_2 * \mathbb{Z}_2$, well, you have the even length words, which give you a circle which is twice as long.

These orbifolds, it's a concept that allows you to deal with all the universal covers and the symmetries of the quotients. So you get some feeling for, you start with a square and then tile by reflection, this gives the Euclidean plane. The new fundamental domain for the torus is four squares. Now we could take the pentagon, and you see, four of them fit together around each corner. Locally it looks like, if you get small you can't see the curvature. The non-Euclidean geometry has flat floor plans with five instead of four walls.

You could do the same thing in 3-space, use the reflections through the faces in 3-space to tile space. In non-Euclidean geometry you replace the squares with pentagons and you get, well, these two sides have to get glued together. This tiles, well, you get a dodecahedron. I didn't finish the previous discussion. Actually, I did. There's a dodecahedron with right angles in 3 dimensions, and you can consider reflecting in its sides, and locally you'll get rooms that look like cubes. You have rooms in this non-Euclidean hotel but now you have twelve walls to cross.

[It seems like the essential thing is to take a platonic solid in Euclidean geometry and put it in non-Euclidean geometry with different angles.]

You have three things coming together at a vertex, so you actually have three,

[distracted by computing angles.]

The non-Euclidean $3D$ crystals are $\mathbb{H}^3/\Gamma$ where Γ is a cocompact torsion free discrete subgroup. So these are, how do you recognize one? Well, π_1 should be infinite, not a nontrivial free product, and should not contain $\mathbb{Z} \oplus \mathbb{Z}$. Also then, Γ should be unique. Here Γ will be the fundamental group and the realization should be unique. The uniqueness is Mostow, and eighty percent of the cases are due to Thurston, with the last parts due to Perelman.

So the vertices can go out to infinity, and the quotient won't be compact, it will have a long snorkel on it like a cusp, so it's not compact but it has finite volume. If you take the cube and pull the vertices to infinity, you finally get these long things, and the angles will be sixty degrees between the sides. You can take a subgroup of finite index with no torsion. So the second type will be discrete cofinite volume torsion free subgroups. The orthogonal thing I was intersecting with to see the small equilateral triangle, it's a sphere with center very far away, a horosphere, and this is a sphere of arbitrarily large radius, so it's the Euclidean geometry.

This will give a tesselation by equilateral triangles, the cross-section at a plane. If you take the right subgroup, you get a little torus. From a topological point of view I'm describing things with torus boundary. Geometrically I'm describing things with finite volume.

In case two the recognition is that π_1 is not a free product, and the only $\mathbb{Z} \oplus \mathbb{Z}$ come from the boundary. All of these boundary π_1 will inject.

Another interesting possibility is to take S^3 minus a knot. Remove an open tube around the knot, and you get as a complement a three-manifold with torus boundary. The figure eight knot, if you do that, will satisfy two. There will be no $\mathbb{Z} + \mathbb{Z}$ except for the boundary. A knot is nontrivial if ond only if the torus' π_1 injects. So there's a real number associated to a knot, the volume.

So these things are the diamonds. I need another set of diamonds, which are the fibered manifolds. Everything else is essentially built from dimension two. One thing you can do is take a circle cross a surface. Take this thing, this disk cross the circle, out and I'll draw it in a funny way. Here, this thing is obtained, and you remove a plug in S^1 cross the surface. Instead of sticking it on by the identity, we'll stick it on by $2\pi iq/p$. Then a point like this, goes over here, on the boundary these are filled with things three times as long as they are high. So you can glue back in this other way. Actually there's now another degree of freedom, you have two families of circles, you identify the base to the base, but as you glue the circles to one another, you could do some rotation. You could do some spinning as you glued things on. You get an integers worth of ways in which a circle can do this, by winding number. You would have a q/p and then also a twist, a twist integer. Then you could do this at any number of places. These are fibered manifolds. One has to take this as a definition. Different twist integers add, they're not independent. You have one global twist integer, fractions, and a surface as a base. These manifolds have an interesting property, if you make the circles short, the volume goes to zero but the curvature does not go to infinity. This is type one of fibered manifolds.

We can do the same construction with surfaces with boundary, and then we get things with boundary. There are one or two examples you have to include, but excluding them, well, the boundary again are tori. If you exclude a small set of examples, with like one or two elements topologically, the boundary injects again. If you take, I'm simplifying slightly for the oriented case, but if you take any path, you have a, well, the circle of the fiber is in the center. So you get a circle bundle over a circle, so in the oriented case this is in a torus. This has millions of $\mathbb{Z} + \mathbb{Z}$. The recognition is that it's not $G_1 * G_2$ but it has a center. The boundary tori in this case ar filled up with circles in the center, so canonically fibered. We could glue two of these things together along the tori, and if you match the fiberings that will be one. That union will again be a diamond. You can break one of these up very carefully. If you glued them incorrectly you'd lose the center.

I'm trying to give you pieces, we'll use them to make jewelry in a second. The next examples will have π_1 finite. The universal cover is compact and simply connected so it's the three-sphere by the Poincaré conjecture. There are five platonic solids in groups of three. Take the finite group of symmetries of one of these things. That sits in the group of rotations, and so you take this as a discrete subgroup. For the cube you get 24 symmetries of the cube, orientation preserving, so you get $S^3/48, S^3/24$, and $S^3/120$. This last one has been constructed before with $3, 4, 5$ fibering over the sphere.

These are all the manifolds we need, and we start making jewelry out of these diamonds. If one of these is closed we don't do anything to it. We can take a lot of these manifolds with torus boundary and we can glue them together along the torus. That's torus-attaching. The map here, I'll say it doesn't preserve the fibering. If it did, it would already be a diamond. So these gluings don't respect the diamond. The diffeomorphisms of the tori are $SL_2(\mathbb{Z})$ as long as you avoid one $\mathbb{Z}$. You can also do self-gluing. We discussed this ad nauseum. This is amalgamated gluing, you also have self-gluing. So you can associate a graph to the gluing pattern. Conversely, you can take any graph and stick in a diamond at each vertex, and then you can glue in a three-manifold.

Now you can make strings of these pearls via two-sphere attaching. So this means you cut a three-ball out of each one and attach them with a little tube. You just make strings, you don't go around in loops.

This is it, these are all the three-manifolds. The diamonds have special geometries. You have $\mathbb{H}^3$, Euclidean space, and S^3. The first set were non-Euclidean space, the platonic ones were spherical. There's also $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$. Then there are the upper triangular matrices with unit diagonal, the solvable group, and $SL_2(\mathbb{R})$. We'll fit these geometries into this picture. Doing the torus gluing forces two geometries together. That's it.