Dennis Seminar
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The symmetrical picture of the exterior algebra is related to exterior algebras. The Lie algebra structure will be related to exterior algebra d. So there's a map δ increasing the exterior power. There are two properties, $\delta(a \wedge b) = \delta a \wedge b - a \wedge \delta b$, with $\delta 1 = 0$ and $\delta^2 = 0$. If just the first nontrivially vanishing composition is zero then it's zero everywhere.

Then there is a theorem that I'm going to make as an exercise.

Exercise 1

1. *δ^2 is zero if and only if that is true from one to three.*
2. *there is a one to one correspondence between isomorphism classes of $(\wedge(V), \delta)$ and Lie algebra structures on V^*. Here δ will be the dual of the bracket.*

This is the tip of a big iceberg.

Here's a geometric interpretation. If you have a Lie algebra with a bracket, a finite dimensional Lie algebra determines a unique simply connected connected Lie group whose tangent space is V at the identity. Then $(\wedge V, \delta)$ is isomorphic to a differential subalgebra of left invariant differential forms on the Lie group.

So three dimensional examples of Lie algebras are in correspondence with three dimensional simply connected Lie groups.

So first, δ might be an isomorphism at level one.

So we have $\delta x = y \wedge z$, $\delta y = x \wedge z$, and $\delta z = x \wedge y$. So this is an example. This is the Lie algebra $so(3)$. This is a $3D$ example. I only know two that satisfy that one hits two like this up to isomorphism. $\delta x = z \wedge x$
$\delta y = z \wedge y$
$deltaz = x \wedge y$

As an exercise check that under this duality, you have these three guys here, the Lie algebra of matrices, well, you have as generators

Exercise 2 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *These are the simple examples. I believe that there are only two in dimension three. When you play around with this, you can see that you can't do much else. Elie Cartan in his thesis classified all of these things, maybe over the complexes. If the δ map is an injection, that basically means the Lie algebra is semisimple.*

So if δ has as kernel containing x.

So $\delta x = 0$. We could say $\delta(y, z)$ is, it's x wedge with some two by two matrix M applied to (y, z). If you differentiate this, you'll get x twice when you differentiate twice. So this is a family of examples for matrices M.

This is a subalgebra, the $\delta x = 0$. So this corresponds to a Lie quotient algebra $\mathbb{R}$. So $A \to S \to R$ is exact, and we'll see that the kernel is a two dimensional Lie algebra, and there are just two of those. If you look at the two-dimensional case. You can put any δ you want. You can either put in zero or an onto map. So one is the Abelian Lie algebra, and the other is the $ax + b$ Lie algebra.

If the kernel is Abelian, well, you have a section, this is a semidirect product of A with R and the action is given by this matrix. I don't want to be exhaustive, I want to generate these matrices. If M is zero you get the Abelian algebra. If M has rank two, well,

- M could have complex eigenvalues, rotation and dilation.
- M real diagonalizable.
- M could be nondiagonalizable.

In the rank one case M could be diagonalizable or not. If it is, you have $\delta x = 0, \delta z = x \wedge z$, and $\delta y = 0$. So you have seperation of variables and get the $(ax + b)$-group $\times\mathbb{R}$. The nondiagonalizable case gives you the Heisenberg group, and the rank zero case is Abelian. I'll keep these three, as well as the two simple ones.

There's the degenerate diagonal case where you have a multiple of the identity (hyperbolic), or a multiple of the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. (solvable). So the other ones don't get kept.

These seven Lie groups with their left-invariant metrics give seven of the eight geometries.

Why aren't the other Lie algebras on the list? Solv has a cocompact discrete subgroup. That is true for the Abelian one as well. There are also some for Heisenberg. Basically everything except the hyperbolic space and the hyperbolic plane cross $\mathbb{R}$. I gave a tiling of the hyperbolic plane that didn't correspond to a discrete subgroup. It was not discrete. If $ax + b$ had one then there would be one for the hyperbolic space too. But you got dyadic rationals. So these two don't have discrete subgroups. You instead, for all of these examples, consider the left-invariant geometries. Take these Lie groups and make them Riemannian. Now we enlarge the Lie group by a compact amount by considering all isometries of this geoemtry. It may be, you can take the full group of isometries, and then you get the full PSl_2 group cross the isometries of the line, and you get discrete subgroups. You do the enlargement and get discrete subgroups. If you had a way to work outside of three-space, you could weld together congruent spherical shells. All the groups are three dimensional, adding one and three to the $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^3$ cases to get things with discrete subgroups.

So Nathaniel was right, there are more candidates for universal covers than there are universal covers. So there are a few extra.

Now I want to do a shape example. Let's talk about the shape or rough geometry of these things. This is a course in algebraic topology. I got the $\delta^2 = 0$. So that was some algebraic geometry. So if we have a compact three-manifold we can take its universal cover and lift a basepoint to get a uniformly distributed subset. So the universal cover has the same shape as the discrete subgroup, which you can think of as $\pi_1(M)$. The word metric takes paths, which is sort of the same as the manifold itself. You can think about the shape of the discrete subgroup. Milnor started by taking the invariant, the growth rate of the fundamental group. Let's do the growth rate of the discrete Heisenberg group. This is the group generated by x and y subject to $[x, y] = x^{-1}y^{-1}xy$ commutes with x and y. Let's count how many group elements there are within various distances, length N.

You can write down a bunch of x and y. Now what we can do in this group, we have a canonical form which is, we can write things, move the x to the front, getting a commutator, and then you move the commutators to the end. So you can get $x^a y^b [x, y]^c$. So a word of length at most N, there are N^2. The best case is when all the x and y are in the right position, so you get about N^2 of the commutators. So the growth rate is approximately N^4. That's the same as the growth of the volume of the universal cover.

I've got to go.