RTG Seminar
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Good afternoon, I'm William Linch, one of three RTG postdocs. Today we'll give a crash course in classical mechanics. I'm going to cover two years in two hours. For the past few weeks Jerry has been lecturing and it's been fairly formal and mathematical. This will be from the physics point of view. There are no questions that are stupid because a lot of it is language. Hopefully not all of it will be.

This is Newtonian, classical mechanics, so let's start with that. We'll start with some number N of particles in d space dimensions. So basically, this will be the simplest possible case, where $N = 1$ and $d = 3$ and will restrict to Euclidean space. The number Nd comes up often in the jargon, and in this context we call it the number of degrees of freedom. In the classical context this is what the number of degrees of freedom means. This is because a point particle has three directions in which it can move. This whole thing I just described is called the configuration space of the system. In this case the system is just one particle.

Next, of course, we want to talk about dynamics, which implies a notion of "time," and we consider particle trajectories, this is essentially the topic under study. These will be represented as maps $t \mapsto x^i(t)$ for $i = 1, 2, 3$. Then there's some path, which is sometimes referred to as the particle's "world-line."

One of the most important quantities you can construct out of this is its time derivative $\dot{x}^i(t) = \frac{dx^i}{dt}$ which is called its velocity, and most of the time we'll want the particle to have mass, and then we have momentum $p^i = m\dot{x}^i$.

[Nate: this is a vector, not a covector?]

Yes. It's just a multiple of velocity so it shouldn't change how it reacts under rotations. Physicists don't know the difference for a while because everything is flat.

We also have the force, a vector, $\vec{F}$, and then Newton's second law is $\vec{F} = \frac{d\vec{p}}{dt}$. If the [unintelligible]is proportional [unintelligible]then we get a more familiar form, $m\vec{a}$.

[Nate: is this a definition of force?]

[Denny: I would say so, except for in a small number of cases.]

So we have $\ddot{x}^i = \frac{F^i}{m}$. So now the only forces we'll care about are the conservative ones. These are ones for which if I have posititions x_i, x_f and a path between them, then I can associate a quantity $W_{fi} = \int_{x_i}^{f_i} \vec{F} d\dot{\vec{x}}$. This should be the difference between the kinetic energies. So conservative means that if $x_i = x_f$ then the work is zero, so $\oint \vec{F} d\dot{\vec{x}} = 0$, so $\vec{\nabla} \times \vec{F} = 0$, so $\vec{F} = \vec{\nabla} V(\vec{x})$, where this is the potential for F. Things like friction which violate this are in some sense not fundamental.

Okay, so now $m\ddot{x}^i + \partial^i V(\vec{x}) = 0$ is Newton's second law, where $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial^i = \delta^{ij}\partial_j$.

It's probably fair to say that this is the simplest nontrivial example. This is the harmonic oscillator. There's a fixed point, a spring, with a particle at the end. If you move the particle from 0 to x then you have a restoring force.

Hooke's law, which is an approximation to a real spring, in $d = 1$, then $F = -kx$. So it's proportional to the displacement and it's restorative, hence the sign. k is the spring constant or the "stiffness." This is the most important example because it's the first term in the Taylor expansion of any force.

All of this stuff has extra structure on it. time has units $[t] = T$, displacement $[x] = L$, and $[m] = M$ for mass. So $\ddot{x}$ has units LT^{-2} and so force has MLT^{-2}. Then $[k] = MT^{-2}$.

Let me now rewrite, the equation now, let me first say that the potential energy is $\frac{k}{2}x^2$. The equation we want to solve is $(\frac{d^2}{dt^2} + \omega_0^2)x(t) = 0$. Then $\omega_0 = \sqrt{\frac{k}{m}}$ so $[\omega_0] = T^{-1}$.

This came from $-kx = m\ddot{x}$ so that $(\frac{d^2}{dt^2} + \omega_0^2)x = 0$, so $V = \frac{x}{2}x^2$.

Then $x(t)$ is harmonic and $x(t) = \int_0^\infty \{A_\omega e^{i\omega t} + A_\omega^* e^{-i\omega t}\} d\omega$, and so

$$0 = \int_0^\infty (-\omega^2 + \omega_0^2) A_\omega e^{i\omega t} d\omega + \{A_\omega e^{i\omega t} + A_\omega^* e^{-i\omega t}\} d\omega$$

plus the hermitian conjugate where $A_\omega = \frac{1}{2}A_0 \delta(\omega - \omega_0)$.

$$\int_0^\infty \frac{1}{2} A_0 \delta(\omega - \omega_0) e^{i\omega t} d\omega$$

plus the hermitian conjugate, and so we get $\frac{1}{2}A_0 e^{i\omega t}$ plus the hermitian conjugate. Then $x_0 = x(0)$, which is just $Re(A_0)$. Then $v_0 = \dot{x}(0) = -\frac{1}{2i}\omega(A_0 - A_0^*)$. Then $p_0 = -\omega_0 m \, Im(A_0)$.

Then the full solution, first, $A_0 = x_0 - i\frac{1}{m\omega_0}p_0$ and the general solution, which we didn't need two boards to derive is $x_0 \cos(\omega_0 t) + \frac{p_0}{m\omega_0} \sin(\omega_0 t)$.

Let me say a little about energy, and then we can get to Lagrangian mechanics. We have kinetic energy $T = \frac{1}{2}m\dot{x}_i\dot{x}^i$, summed over i. The statement is that if there is no external force then this is constant in time, so $\dot{T} = m\ddot{x}^i x_i = F x_i$. Then the potential energy is $V(\vec{x})$ so $\vec{F} = -\nabla V$.

[Nate: what is x_i?]

It's just a lowering of the index with the metric, $g_{ij}x^j$, where here $g_{ij} = \delta_{ij}$. Now we can construct the total energy $E = T + V$ and it is always conserved, as

$$\dot{E} = \dot{T} + \dot{V} = m\ddot{x}^i \dot{x}_i + \frac{\delta V}{\delta x^i}\dot{x}^i = (m\ddot{x}^i + \partial^i V)\dot{x}_i = 0$$

by Newton's second law.

What we call the ground state is the minimum of E. This requires $T = 0$. Here $x^i(t) = x^i_*$ and $\dot{x}_* = 0$, $\partial_i V(x(t))|_{x(t)=x_*} = 0$. The ground state of the harmonic oscillator is $x_* = 0$. So we shift the potential energy of the ground state to call this zero, as the potential was only defined up to an additive constant.

0.1 Lagrangian formalism

Okay, now Lagrangian mechanics, the Lagrangian formalism. So we, the trajectory has a tendency to lower the potential energy. You might say, that's very convenient. Once you put kinetic energy into the problem, things aren't so simple, you can't just minimize. So if you include the kinetic energy, can you find a scalar quantity depending on possible trajectories $\vec{x}(t)$ whose minima coincide with the physical trajectories? These are paths that solve Newton's second law. You call this "action." This is based on, trajectories try to minimize potential energy. The paths that minimize potential energy are the ground states, we want extrema on all physical trajectories. So it's natural to start with a V, but that can't be everything, and we know $E = T + V$ is conserved, so let's try the difference, $L = T - V$. This is not the function or scalar quantity, because it's not constant over the path.

So $L(\vec{x}(t), \dot{\vec{x}}(t)) = T(\dot{\vec{x}}(t)) - V(\vec{x}(t))$. This is a function of time. We don't want that, so integrate it, $\int_{t_i}^{t_f} L(x(t), \dot{x}(t))dt$. It's natural when I study a mechanics problem, to have something that starts at one time and ends at another. We call this $S[\vec{x}(t)]$, the action. So why do we invent this thing, or postulate it, or try this?

If we make variations of the path and this is stationary, it should satisfy Newton's second law.

Take some purported reference path $\vec{x}(t)$, and consider a path that is close to it in some ϵ-sense, some tubular neighborhood, the displacement, $\vec{x}(t) = \vec{\bar{x}}(t) + \epsilon(x(t))$. Call ϵ δx. So $\delta x(t_f) = 0$ and $\delta x(t_i) = 0$. I'll take the linear part of x evaluated on this whole thing, the part which is the ϵ. So I take $S[x^i] = S[\bar{x}^i] + \delta S[\bar{x}]$. So

$$\delta S = \delta \int_i^f dt L(x, \dot{x}) = \int_i^f dt \delta L(x, \dot{x})$$

$$= \int_i^f dt\{\frac{\partial L}{\partial x^i}\delta x^i(t) + \frac{\partial L}{\partial x^i}\underbrace{\delta\dot{x}(t)}_{\delta\frac{d}{dt}(x(t))=\frac{d}{dt}(\delta x(t))}\}$$

$$= \underbrace{\int_{t_i}^{t_f} dt\{\frac{\partial L}{\partial x^i} - \frac{d}{dt}(\frac{\partial L}{\partial x^i})\}\delta x^i(t)}_{\text{Euler Lagrange Equation}} + \frac{\partial L}{\partial x^i}\delta\dot{x}(t)|_{t_i}^{t_f}.$$

This second term I'm going to ignore. Jerry is going to use it. The Euler Lagrange equation is then $\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x^i} = 0$.

Now let's go back to the oscillator. Remember $L = T(\dot{x}) - V(x)$, with $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$. Then $\frac{\partial L}{\partial \dot{x}^i} = p_i$ and $\frac{\partial L}{\partial x^i} = F_i$. Now we have covectors instead of vectors. This is not grad anymore, it's the exterior derivative, $F = -dV$. These are both forms, momentum and force. So we can then apply this to the Euler Lagrange equations and get $\frac{dp_i}{dt} = \frac{\partial L}{\partial x^i} = F_i$.

It's pointless to write the action, but just so that you can see it,

$$S[x(t)] = \int_{t_i}^{t_f} dt\{x\frac{d^2}{dt^2}x + \omega_0^2 x^2\}.$$

I should say $[S] = ML^2T^{-1} = (M\frac{L}{T})L$, the units of angular momentum.

$\ddot{x} + \omega_0^2 x = 0$ are the Euler Lagrange equations here.

[Dennis: do you get a well-defined mechanics with a manifold and a function on the space of paths?]