# RTG Seminar
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Mondays are my worst days. They're over at five, but they start at eight thirty.

[It's okay, you're still young.]

Maybe I'll start by reminding you what I did last time. Last time we constructed the action and equations of motion for a charged scalar field. We showed how it was related to electromagnetism. I want to briefly discuss an $SU(2)$ gauge field. So it's

$$\int -\frac{1}{4e^2} Tr\ F_{\mu\nu}(A)F_{\mu\nu}(A) = \int -\frac{1}{4e^2}|F|^2,$$

where

$$F_{\mu\nu} = \delta_\mu A_\nu - \delta_\nu A_\mu + [A_\mu, A_\nu],$$

or $F = dA + A \times A$. When I need the signs I'll put them in carefully. You can also write $|F|^2$ as $tr\ F \wedge *F$. So that's the gauge part and now I have, well, in the previous case I had $\mathbb{C}$ as a vector space, and now we'll have a three dimensional real vector space. So we get in the integral $\frac{1}{2}\nabla^\mu\Phi\nabla_\mu\Phi + V(\Phi)$. Here $\nabla_\mu\Phi = \partial_\mu\Phi + [A_\mu, \Phi]$. This is bad, I'm assuming that I'm working in the adjoint representation so that $\Phi$ is a traceless two by two anti-hermitian matrix. So if $\nabla\Phi = d\Phi + [A, \Phi]$, I can write $\frac{1}{2}|\nabla\Phi|^2 = \frac{1}{2}tr\ \nabla\Phi \wedge *\nabla\Phi$.

So then we studied the variational problem that came from this. So $\frac{\partial S}{\partial A_\mu} = 0$ gave rise to $D^\nu F_{\mu\nu}(A)$ up to sign being equal to $[\Phi, \nabla_\mu\Phi]$.

The other equation we got was from $\frac{\partial S}{\partial \Phi} = 0$, which gives us an equation of the form $\nabla^\mu\nabla_\mu\Phi - \frac{\partial V}{\partial \Phi} = 0$. These are the equations we had last time.

[What is $\Phi$?]

It's a function to a vector space, like a section of an associated bundle.

In general, the commutator would be replaced by $\langle \bar{\phi}T, \nabla\phi\rangle - \langle\bar{\phi}, T\phi\rangle$.

[What is $A$?]

It's a connection, it's a one-form with values in the endomorphisms of a representation space.

Okay, so the first thing I want to consider is the Higgs mechanism, which for physicists is simple but for mathematicians is mysterious. Let me first do the following. If I take $V$ to be $\frac{\lambda}{4}(tr\ \Phi^2 - (\mu I)^2)^2$, then this is the famous Mexican potential, with the shape of an upside down sombrero. I'm trying to get you a picture af what's going on.

So I can choose any point along the two-sphere that sits down at the minimum, well, I want to look for solutions to these equations. I want to give these things values. I'm going to look for the simplest kind, where $A$ is zero and $\Phi$ is a constant. So I choose a value for $\Phi$, so in particular that $\langle\Phi\rangle$ is nonzero, something like $\mu \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right.$

In particular, I'm going to change the potential in a mament. This will give us boundary conditions at $\infty$. Now the interesting feature of this is if I look at values of the Lie algebra under a gauge transformation, so thta if $\lambda$ is in the Lie algebra, we write $\delta_\lambda \Phi$ is just $[A, \Phi]$ up to sign. Now I can ask for the stability subgroup, which is $\Lambda$ proportional to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\partial_X \langle\Phi\rangle = 0$ implies $[\Lambda_1, \langle\Phi\rangle] = 0$.

This is the Lie algebra corresponding to the stabilier subgroup.

[I left the room briefly.]

So that means that the solutions will have very different properties. The wave solutions will be massive. I can't do gauge transformations any more because those change $\Phi$. The only ones I'm allowed to do are the $U(1)$ ones, because they preserve $\partial_\Lambda \langle\Phi\rangle = 0$.

You get $\Phi$ which commute with the expectation value and have no charge, and then ones which don't commute and don't have charge. We have an $SU(2)$ bundle and an associated bundle and then reduce that to an $S(1)$ bundle.

There is another interesting aspect of this.

[How does this massive term arise?]

I can write $|\nabla\varphi + [A, \langle\Phi\rangle]|^2$ which will have three terms,

$$\frac{1}{2}|\nabla varphi|^2 + tr\ \nabla\varphi[A, \langle\Phi\rangle] + |[A, \langle\Phi\rangle]|^2.$$

All three of these are the same, but this is a special case. We have two masses, $\mu^2$ and 0.

[You're looking at the first problem, you've chosen a minimum, and we're studing the space around the minimum to see what the action gives us, and now you get something like last week, a scalar filed with masses in front of it, and the non-Abelian gauge group is now gone.]

This is called the Higgs mechanism. I want to tell you two more things about it. Well, $\partial\varphi = [\Lambda, \varphi] + [\Lambda, \langle\Phi\rangle]$. The second term vanishes in the $U(1)$ case. But in general I have a nonlinear representation. This suggests that it is correct to choose a gauge to get rid of two of the $\varphi$, to set them to zero. I can choose $\varphi$ to just have the form as a matrix $\begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix}$ because I can just choose my parameters to get rid of the other terms.

If you look at the reperesentation theory of the Poincaré group, the massless representation has two states for a gauge connection one-form. Being massless you move in some direction ot the speeed of light. The states are the spin. Massive particles have three states, you can bring them to rest and so on. The connections corresponding to the broken directions beocomes' massive.

The final thing that magically works together is if I compute the mass matrix, the Hessian, the mass matrix here, I get only, there's a, the directions in the minimal are massless and get eaten. The uphill ones are the mass.

Okay, so this is the background. Please, if there are more questions ask them now. The basic thing I want you to take away from this, I want next to look at some different kinds of potential.

[We still have the crassterm that will look at $\varphi$ and $A$, so you left something over.]

I don't remember, these might vanish when I do things carfully.

I should say, I could have used many copies of representations. Whether I have isotropy defends on the details.

Before the symmetry breaking I had $A_n$ and $\Phi$. These are quadratic term, every term has a derivative. After symmetry breaking, we have $A_\mu$ and $B_\mu$, where $m = 0$ and $m = \mu$ respectively. So in the first series we get $2 \times 3$ and 3 is nine. In the second we get two, two times three, and one.

This is nothing more than a rearrangement, a change of basis or coordinates. In the original thing there are hills, we don't know it while we're in it.

I want to put the mass in by hand by saying it's the boundary condition. You can do the next part of the talk,

The one change I want to make in this whole story, is, I'm going to take two $\Phi$, and I'm going to choose my potential to have the following simple form $\frac{1}{2}tr\ ([\Phi, \bar{Phi}]^2)$. Okay? So why is this related? The reason it's related is because there's no $\mu$ there. If you look at the function it's semidefite, bounded below, and has valleys when $\Phi$, $\bar{\Phi}$ commute. Th vacuum is described by $\langle\Phi\rangle, \langle\bar{\Phi}\rangle = 0$. I can always choose $\langle\Phi\rangle = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\bar{\Phi}$ having the same except with $\bar{a}$.

[I'd like to understand where the boundary conditions are put in.]

An answer?

Now I'm going to do something. I'm not sure. So I don't know how much, I'm going to restrict now to static $\Phi$. I will assume they are time independent. I can replace $\mu$ with $i$ in my indices. Now I'll be careful with my factors. We have our potential, and we'll rewrite the first term of the integral into static $\Phi$, the potential which stays the same, and the middle party. Well, I'll assume $[\Phi, \bar{\Phi}] = 0$ so $V(\Phi) = 0$. Let $E_i = F_{0i}$ and thnen $B_i = \frac{1}{2}\epsilon ijkF_{jk}$. Now I rewrite this as follows. I have the imaginary part, no. The real part I actually want. This can be wriin as $B^2 - E^2$. We will rewrite all of this now as

$$H \sim \frac{1}{2}(B + iE + sqrt2\nabla\phi(T))(B - iE + \sqrt{2}\nabla\bar{\phi}) + \text{surface term}$$

This surface term works out, we don't have to worry about that, a lot of these things give surface terms we don't have to care about. Now I want to compute the Hamiltonian instead of the Lagrangian. The term with time derivatives changes sign. We take minus the potential, and if $p$ is $\dot{q}$, when you plug this in, you switch the sign of the $E$ terms. The $B$ terms sit in the potential, they're spatial. The $E$ term has a time derivative so change sign. So that's why the $E$ becomes a plus. You change to the Hamiltonian because there are states of lowest energy. For the cases we were looking for here, we were just looking at the potential.

The statement, $B^2 - E^2$ is a Lorentz invariant. The $B^2 + E^2$ is not supposed to be an invariant. Since everything is staggered we have a preferred Lorentz frame, we're looking for a ground state, minimizing the energy. So now we can look at the variation of the equation above. The surface terms are held constant under the variations. So the equation we're going to look at (I could also introduce a phase and the conjugate phase), here $\Phi$ is purely magnetic. So I'm going to choose to solve the following equation. The term we have is an absolute value squared. The minimum will occur when this thing vanishes, when I get a first order equation. This is consistent with the second order equations I started with, it's like a square root, it's stronger, it implies them.

So let's see what you get. My equations look like

$$B + iE + \sqrt{2}\Phi = 0, \quad B - iE + \sqrt{2}\bar{\Phi} = 0$$

I want to look for very definite solutions. So there's a funny story, the basic idea is that we want a solution obeying the equation $\langle tr\ \Phi^2 \rangle = a^2$ so that there is a nontrivial knot in the solution. We take out the magnitude, and we have $S^2$ and a map of this $S^2$ to the $S^2$ at infinity. We have this map of the two sphere and it may have nontrivial wrapping number. That is exactly the surface term, scaled by something. We're going to look for a solution of the following type. Let $A$ range over $1, 2, 3$, a Lie algebra valued index. So let $\varphi^A - e^A\varphi(r)$, where $e^A$ points out from the origin. This is an ansatz, which is just a guess, you make a guess and then it simplifies the equations and you solve the remaining equations more easily. You also have the boundary condition that $\varphi(r \to \infty) = a$. Furthermore we make the ansatz that $A_0^A = e^A b(r)$ and $A_i^A = \epsilon_{ij}^A e^j \left(\frac{1-L(r)}{r}\right)$. All of these $e$ are not exponentials but basis elements. Maybe I should write this $\hat{e}$.

So now we just take this and plug it in. We can plug this in and get equations which can be integrated explicitly. I chose, I had everything worked out and so I used our paper, but we didn't work everything out. Okay, so $L$ has the form, well, let me write down the equations.

$$\frac{d^2}{dr^2}\log(L) = \frac{L^2-1}{r^2},$$

but if we substitute $L = re^{\rho}$, we get, well, let me say,

$$B_i^A = \hat{e}_i\hat{e}^A\frac{L^2-1}{r^2} + \Pi_i^A\frac{L_r}{r}$$

where $\Pi_i^A = \delta_i^A - \hat{e}^A\hat{e}_i$. Now

$$E_i^A = -\hat{e}_i\hat{e}^A - \Pi_i^A\frac{bL}{r}$$

$$\sqrt{2}\varphi_r = \frac{1-L^2}{r^2} + ib_r, \quad \sqrt{2}\varphi = \frac{-d}{dr}(\log\ L) + ib$$

The solution I get when I do this, I get $L = \frac{\kappa r}{\sinh(\kappa(r+\delta))}$. Now we can find the other things. So we can compute, $L$ is real, so we can get the real and imaginary parts of $\varphi$ and just compute it from this. We find that the real part of $\varphi$ is $\frac{\kappa}{\sqrt{2}}(\coth(\kappa(r+\delta)) - \frac{1}{\kappa r})$. So we want to get regularity at $r = 0$, which forces something like $\delta = 0, \kappa = 1$.

These solutions are extremely interesting. There is a region where, if I look at $||\Phi|^2 - |A|^2|$, what happens? At $\infty$ it is zero. There's a localized lump near the origin. If we compute the magnetic charge of this, it will be a magnetic monopole, it has a magnetic charge. We have a $U(1)$, so it makes sense to talk about the magnetic charge. It's a different gauge than earlier but that's okay. You can ask about the electric and magnetic charge at $\infty$. You get a magnetic monopole that you can't get rid of. Normally in electromagnetism, the Bianchi identity tells you that there are no magnetic monopoles. But in the core region you can violate that.

So there should be traces everywhere. This a non-Abelian electromagnetism, which at $\infty$ gives the Abelian electromagnetism.

[You said earlier that $B$ and $E$ don't make sense. Is their separation gauge invariant?]

Yes. Gauge rotations mix the $E_i$ into each other and the $B_i$ into each other Only Lorentz transformations mix the two. I recommend the undergraduate text on physics by Purcell for a beautiful description of how this can all be viewed as consequences of the Coloumb force.

The whole point of gauge theory is that I can apply a group action at every point in a contractible set at the same time. If I try to extend this globally I get what looks like a Dirac monopole. The statement that the divergence of $B$ is zero implies no monopoles is only true on a simply connected set. So you can write a solution for a solenoid. We can imagine the limit of an infinitely long, infinitely thin solenoid. That will look like a monopole, except if you hit the solenoid, the line. If the monopole obeys a certain condition, then even the solenoid will not be detectable. The gauge potentials are singular but the electric and magnetic fields will still be well-defined.

You try to make $\Phi$ parallel everywhere, and you will find a Dirac string pointing along.

[You're trying to trivialize a nontrivial bundle over $S^2$.]

Right, if I do that I get a singularity.

[Do you have to trivialize it?]

If you have different elements along the sphere at infinity, that's fine.

Mathematically this is a very beautiful object. These first order equations have very nice properties.

[What do these represent?]

If you compute the energy or the mass of the monopole, you have this condition, you find that the charge of a monopole is inverse to the charge of an electron. Now electrons have very small charge. Also, the mass grows with the charge. So if electrons have small charge, monopoles have big charge, so they're very massive. There are grand unified models that have things like this, where you can have monopole solutions. They would be produced in the early universe, in the big bang. People have done searches, and it's unresolved whether these things exist or not.

[Does $\Phi$ have any meaning?]

The Higgs field is an example. That's not in the adjoint representation but it's the same kind of structure.

[How specific is this monopole picture to the Lagrangian you started off with?]

I can write down many other gauge groups and representations. Any time you have gauge groups with $U(1)$ isotropy arising you get one. You can make it a precise homotopy question.

[Can you get $\Phi$ as a reduction of a pure gauge theory by dimension reduction?]

Absolutely. For example, when you compactify string theory on [unintelligible].

So there are plenty of examples.

[Looking for the Higgs boson, so, is that described as a monopole?]

No, it's not in a representation that gives you nontrivial isotropy. This breaks the $SU(2) \times U(1)$. You have isotropy but not the right kind.

[Is this how this mechanism arose?]

It was first done in a $U(1)$ model. The history went as follows. People realized you could write down symmetries that are broken. You had these flat directions which gave you massless particles which weren't observed. Then Higgs and a whole bunch of other people realized that you get the Higgs mechanism where the connections absorb these and become massive vectors. This was understood as a general mechanism, not as applied to a particular symmetry.

There's one more part of the story that is really interesting called $S$-duality. If you look at electromagnetism, there is a symmetry between the electric and magnetic equations. If you put matter in, charges, you break this because you don't have magnetic monopoles. But in this non-Abelian context, you have more hope of dualizing, and for $n = 4$ supersymmetric Yang Mills theory, there is a duality, namely, even in the presence of monopoles. All the evidence you have says that the theory with a certain coupling, with electric charges, is entirely equivalent to the theory with the inverse coupling and magnetic charges.

One problem with monopoles is that if electrons are coupled weakly then monopoles are coupled strongly, and now strong coupling becomes weak coupling. This notion of duality in string theory gives us all kinds of relations between theories, and is also central to Seiberg-Witten.

This means something different to mathematicians and physicists. This is supposed to be equivalent to (non-Abelian) Donaldson, but it's Abelian. It differs in emphasis. You want to use it to understand 4-manifolds. The physicists want to understand the gauge groups.

[Mathematically what algorithm do you perform to talk about particles?]

You want to do that here because it's massive, with one unit of magnetic charge and zero units of electric charge. Then you can look at the energy density, which is localized in a compact region. Then it makes sense to think of it as a localized object, you can kick it, apply forces to it, it follows Newton's laws, or the relative versions. It's very important that it's a stable solution that doesn't disperse, or else you wouldn't consider it a particle.

[There was this illegal conversation going on, saying the words boson, counting degrees of freedom. I wanted you to say, look at this, it's the spectrum of some operator, and get the particles out of somewhere.]

I haven't talked about the spin of a monopole. I looked at the symbol of the differential equations, look at the top level thing, and then I get a solution. At the classical level I can just deal with bosons. This monopole is nonperturbative. It's not a small fluctuation from the vaccuum. If I now, if I try to write down an equation to govern how it moves, how would I apply a force to it? I would turn on a magnetic field, it could do all kinds of things. It's not a pointlike object. But if you have this duality you can do it. For suitably strong enough coupling, it becomes so heavy, I don't know. Well, it becomes very compact and acts like a particle.

[What about with many monopoles?]

There you have more parameters. Also, this is in the moduli space, which ignores this internal structure.