MAT :Algebra III
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I want to put forward another lemma, about the intersection of primes in an ideal.

Proposition 1 *Let $p_1, \dots, p_n$ be prime ideals in a commutative ring A, and I an ideal. If $I \subset \cup_{i=1}^n p_i$, then $I \in p_i$ for some i.*
If $I_1, \dots, I_r$ are ideals and p a prime ideal, and $p \supset \cap_{i=1}^r I_i$, then $p \supset I_j$ for some j. Furthermore, if $p = \cap_{i=1}^r I_i$, then $p = I_j$ for some j.

The proof is by induction. This is true for $n = 1$, yes? There's nothing to prove. So what do I want to show? If an ideal is not contained in any p_i, then it is not contained in their union. So for every i, there exists an $x_i \in I$ by the induction hypothesis, with $x_i \notin p_j$ for $j \neq i$. So if $x_i \notin p_i$ for some i, then we're done. So assume $x_i \in p_i$. Then $\sum_{j=1}^n \frac{\prod_{i=1}^n x_i}{x_j}$ is not in p_i for any i. This is because the product of elements not in p_i is also not in p_i.

Keep in mind this trick; it may prove beneficial. Now, how do you prove the second one? Prove the second part of the proposition. You don't even need induction. If p doesn't contain any of the ideals, then there is an element not in p in each ideal; then the product is not in p.

Questions with what we've done up to now, or should we go on to something new? I'm going to try to convince you with a very concrete problem.

Now let $S = k[x_1, \dots, x_n]$. We know about $k[x]$ that it is Euclidean, PID, UFD, noetherian. What survives in multiple variables? Let's look at the Euclidean algorithm. This says that if you have two polynomials f, g that you can write $f = gh + r$ "uniquely" with r of smaller degree. This shows that you are a PID by taking the lowest degree remainder. You don't get division with more than one variable. You have a total order by degree in one variable. In two variables one doesn't know how to order them. Which is bigger, x_1x_2 or x_1^2?

Now we're looking here because the zero loci describe nice sets. If you look to noetherian rings, every one which is a k-algebra is a polynomial ring modulo an ideal. So that's why we're interested.

Now in $k[x]$ the actors were just variables raised to powers. What is an ideal in $k[x_1, \ldots, x_n]$? There will be two reductions. One is computational. The other will be a division of some sort. So the ideals. The simplest ideals in $k[x]$ are those generated by a power of a variable, x^n. So in the multivariable case, the simplest ideals are the monomial ideals $I = \langle \underline{x}^{\underline{\alpha}} | \alpha \in A \subset \mathbb{N}^R \rangle$. Here $\underline{\alpha} = (\alpha_1, \ldots, \alpha_R) \in \mathbb{N}^R$; $\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1} \cdots x_R^{\alpha_R}$.

Now look at a basis of $k[x_1, \ldots, x_R]/I$ as a k-vector space. This will be all monomials not in I. So, we can see that $\underline{x}^{\underline{\beta}} \in I \iff \exists \underline{\alpha} \in A$ such that $\underline{x}^{\underline{\alpha}} | \underline{x}^{\underline{\beta}}$, i.e., that $\underline{\beta} - \underline{\alpha} \in \mathbb{N}^R$.

Look in two variables, $I \subset k[x, y]$. Let $I = (x^3y, x^2y^2)$. You draw these by plotting the exponents. So we plot the points $(3,1)$ and $(2,2)$ on a planar graph. Then anything above and to the right of either of these points is in the ideal. It has hidden corners, such as $(3,2)$.

This is not Artinian, i.e., it doesn't have the descending chain condition. So look at $k[x, y]/I$. To make this finite dimensional, you need some power of each of the variables. So the quotient of a polynomial ring by a monomial ideal is finite dimensional if and only if it contains a power of each of the variables. That's what we just proved with the picture.

Now look at the ideal $J = (x^5, x^3y, x^2y^2, y^6)$. This is Artinian and we can count the dimension. The corners are $(2,6), (3,2), (5,1)$. These have a meaning. This will be related to the primary decomposition.

Lots of information is in this picture, for instance the number of elements in the convex hull not in the ideal. Anyway, in this case we count off lattice points and see that the dimension is 16.

There is a little more geometrically behind monomial ideals. I'm going to make a detour before telling you about it. This will let you prove the basis theorem constructively. Some mathematicians were very upset with the nonconstructive proof of the basis theorem, and we'll follow their methods.

Let's look at simplicial complexes.

Definition 1 *Δ is a simplicial complex with vertex set $R = 1, \ldots, r$ if*

- $\Delta \subset 2^R$.
- $\emptyset \in \Delta$
- *$F \in \Delta$ implies that every subset of F is in Δ.*

So triangulations are simplicial complexes, for instance $2^{\{1,2,3\}} \backslash 1, 2, 3$ which encodes the circle. We call a subset in Δ a face. Or we could write S^2 as a hollow tetrahedron, i.e., $2^{\{1,2,3,4\}} \backslash \{1, 2, 3, 4\}$. But we can also have triangulations without well-defined dimension, such as $\{1, 2, 3, 4, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}$, which is a disc wedge a circle [sic] and not a manifold.

Most of the topology you can see in simplicial complexes can be seen in the algebra. Let's talk about the base ring of a simplicial complex on the vertex set $\{1, \ldots, r\}$. We place the vertex set in correspondence with a set of variables x_i. We define the ring $k[\Delta]$ to be $k[x_1, \ldots, x_r]/I_\Delta$. We define I_Δ as $\langle \underline{x}^{\underline{F}} | \underline{F}$ is not a face in $\Delta\}$. So if $F = \{1,2\} \notin \Delta$, then $\underline{x}^{\underline{F}} = x_1x_2$. To every ideal generated by square free monomial ideals corresponds a simplicial complex, so they're in bijection. This is due to Stanley and Reisner. Almost all of the stuff you want in topology can come from this.

The generators are the minimal nonfaces (a nonface is something that is not a face. Sorry, but combinatoricists like such nonwords). If I look at $V(I_\Delta)$ I'll get a linear realization of Δ, where we extend each face to a subspace.

So for the circle, the ideal is $(x_1x_2x_3)$; for the sphere it is $(x_1x_2x_3x_4)$. For the third case, the ideal is $(x_1x_4, x_2x_3x_4)$.

Let's do one example. Say that the simplicial complex is generated by $\{1,3\}, \{2,4\}$. Then what is $k[\Delta]$? Well, I_δ is $(x_1x_2, x_1x_4, x_2x_3, x_3x_4)$. Now we can also find a basis. We have $f_\Delta = (f_0, \ldots, f_d)$, the f-vector of Δ. Here d is the dimension of Δ and f_i is the number of i-faces, i.e., faces supported on $i+1$ vertices. The Euler characteristic χ_Δ is the alternating sum $\sum_{i=0}^{d}(-1)^i f_i$.

So what is a basis of $k[\Delta]$? Since a square-free monomial ideal is homogeneous, this is a graded vector space, i.e., decomposes degree by degree. As vector spaces (not rings) we have $k[\Delta] = k[x_1, \ldots, x_r]/I_\Delta = \oplus_{i=0}^{\infty} k[x_1, \ldots, x_r]/(I_\Delta)_i$.

Lemma 1 *A k-basis for $(k[x_1, \ldots, x_r]/I_\Delta)_m = k[x_1, \ldots, x_r]/(I_\Delta)_m$ is formed by all monomials $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ with*

- $\sum_{i=1}^{r} \alpha_i = m$
- *the collection of indices with $\alpha_i > 0$ is a face of Δ.*

The proof is obvious.
So what is the dimension of that? I'm going to define a function $H(k[\Delta], n)$. What is this dimension? It's $\sum_{i=0}^{d} \binom{n-1}{i} f_i$. This will turn out to be a Hilbert function.

I want to go over the correspondence.

Theorem 1 *There is a bijection between simplicial complexes on $\{1, \ldots, r\}$ and proper square-free monomial ideals of $k[x_1, \ldots, x_r]$.*

To go backward, look to the elements of the basis which survive, and those give you back the basis. It goes through the exterior algebra.