ALGEBRA III
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Let's get back to localization. Let $S \subset R$ be a muliplicative system. Then $R \to^{\varphi} S^{-1}R$. Here $\ker \varphi$ is $\{r \in R | \exists s \in S, sr = 0$.

So for an R-module M, we can do $M \to^{\varphi} S^{-1}M = \{(m, s) | m \in M, s \in S\} / \sim$. This is an R-module.

What we're going to prove is very easy. There's a natural map: $S^{-1}R \otimes_R M \to^{\sim} S^{-1}M$ as $S^{-1}R$-modules. This will also show that localization is a flat functor. I still want to put a couple of examples.

Example 1. *I said that the standard examples were the following.*

(1) $S = R \backslash$*(zero divisors). This is the largest localization such that* $R \hookrightarrow S^{-1}R$.
(2) $S = R \backslash p$.
(3) $S = \{f^n : n \geq 0\}$.

The first is interesting for algebra; the second and third are geometric. Let me convince you.

So say I have a point $p \in X$, for k an algebraically closed field, which for today we will call $\mathbb{C}$. So I have $p \in X \subset k^n$, an affine algebraic set (i.e., the locus of an ideal). Then $X = V(I), I = \sqrt{I} \subset k[x_1, \ldots, x_n]$.

So I want to know what happens nearby, say, a point. What are the open sets containing p? The Zariski topology has as closed sets the varieties of ideals; then the open sets are complements. So $p \in X \backslash Y$, for $Y \subset k^n$ an algebraic set.

So the larger Y is, the smaller the neighborhood, no? So please keep in mind that the neighborhoods will never be small; the topology is quite coarse. The largest Y would be to take Y as the zero locus of a single polynomial f. You'd better assume that $f(p) \neq 0$ so that $p \in X \backslash Y$.

So if $q \in X \backslash Y$, then $f(q) \neq 0$ so $r(q)f(q) = 1$ for some $r(q)$. What sense can you make of the function r? It is, where defined, $1/f$.

Now I want to look to a new ideal J, but with what I see here, a new variable times f is 1. So this is $I + (zf - 1) \subset k[x_1, \ldots, x_n, z]$. Now I want to look to $Z = V(J) \subset k^{n+1}$.

So if I project this set onto the first n variables, then this will map onto $V(I) \backslash \{f = 0\}$. And this map is injective. My claim is that this map is continuous in the Zariski topology. It's very easy to check this. Also the inverse is continuous. So I can think of $V(I) \backslash \{f = 0\}$ is the same as $V(J)$.

So to such a neighborhood, $X\backslash\{f = 0\} \equiv V(J)$ has affine coordinate ring $k[x_1, \ldots, x_n]/(I + (zf - 1))$. This is isomorphic to $(k[x_1, \ldots, x_n]/I)[z]/(zf - 1)$.

Now let's prove a lemma. The lemma says the following:

Lemma 1. *Say $f \in R$ is not a zero divisor. Then $R[z]/(zf - 1)$ is isomorphic to $S^{-1}R$ where $S = \{f^i, i \geq 0\}$.*

So I don't want any factor of f to factor completely on X. If it is a zero divisor mod X, then some factor of f vanishes on a component of X.

So this is $S^{-1}(k[x_1, \ldots, x_n]/I)$. So the complements of hyperplanes are these coordinate rings. And such complements form a basis of the Zariski topology.

So why is the lemma true? We have to build a map. I'll build first backward. You can go $R \to R[z]/(zf - 1)$. You can do this by $x \to \bar{x}$. When does a map here induce a map to the fractions? It maps to $S^{-1}R$, so when does it induce a map? I need that every element of S is invertible in $R[z]/(zf - 1)$.

Now, S consists of the powers of f. If f is invertible then so is f^n. So $\bar{f}\bar{z} = 1$, so there is an induced map. You can get by with nilpotents, but a nilpotent element eventually vanishes nowhere. You can write this a bit more general.

Now what I'm saying is that this map is an isomorphism. Anything in $R[z]/(fz-1)$ is the class of a polynomial in $R[z]$. I replace z with $1/f$. So $\alpha = P(\bar{z}) = P(1/f) = P(\bar{f})/(f^{deg\ P})$. So this is an element of $S^{-1}R$ which maps to P.

So the thing to show is that it's injective. Then you need nonnilpotence. If you have something equal to zero, then it's a combination of $(zf - 1)$ and you'll eventually get a nilpotent.

I have to show that something that $S^{-1}R$ can't map to a multiple of $zf - 1$. You identify variables. Anyway, it's extremely useful.

Ah! Let's try to do a picture. Let's see if I manage to do something. Say X is the complex line and Y is $\{0\}$. What is $V(J)$? It sits in $\mathbb{C}^2$. Here I is 0. Then this is $zf - 1$. So this is $V(zx_1 - 1)$. If you draw the picture in the reals, it is the graph of $1/x$, a hyperbola. But you do the same thing over the complex numbers. Things get bad if I go into k^2 here.

If I look to the complement of the origin in $\mathbb{C}_2$, that's no longer a hypersurface, so you won't get an algebraic set. It's still Zariski open but it's not as easy as this one. So localization at f is just kicking out $f = 0$ and looking to what's left.

The other one is a bit more complicated, so let's try to disentangle it. I said choose $p \subset R$ a prime ideal. It took many years till Grothiendiek came, to tell what this means. You need to get into schemes.

So one case is easy, which is when p is a maximal ideal. We want to look at $R\backslash p$. We call $S^{-1}R = R_p$. You're localizing at a prime. This gives you a local ring, i.e., one maximal ideal. You have a map $R \to R_p$. So look at pR_p. This is a maximal ideal, and the only one, in R_p.

How do you show this? How do you check if a ring is local? It's equivalent to show that everything not in pR_p is invertible. Then every ideal sits in this one. If something doesn't lie in pR_p, it looks like r/s, with neither one in p. So we can multiply by s/r to get to 1.

Since it's a local ring it comes with a preferred field, called the residual field at p, namely $k(p) = R_p/pR_p$. As a very interesting (or trivial) case, if R is a domain, I localize at 0 and get $R_{(0)}, k_{(0)}$, and $Q(R)$. These are all the same thing. Play with this. Everything coincides. So $R_{(0)}$ is already a field, and you're dividing by 0.

The other interesting case is the following. So let's say R is $k[x_1, \ldots, x_n]/I(X)$. So $X \subset \mathbb{C}^n$. If I need it to be, I'll make it prime by making this an algebraic set.

So what are the maximal ideals here? They're maximal ideals containing I. With the Hilbert Nullstellensatz, these will be ideals of points containing I. With inclusion reversal, these are points in X. So maximal ideals in R are $m_p = (x_1 - a_1, \ldots, x_n - a_n)/I$ for $p = (a_1, \ldots, a_n) \in X$. So imagine for a moment that $p = 0$. What does it mean for something to be a polynomial to be in m_p? If I evaluate at 0 it vanishes. If it vanishes then it turns out to be in m_p.

This even works over finite fields. You can contract instead of taking derivatives to make sure you don't run into problems.

So these are polynomials vanishing at p modulo polynomials vanishing over I.

So localize R at m_p. What are these elements? They are fractions f/g (modulo I, but I'll abuse notation and ignore things that vanish on X) Here f is a polynomial and g is a polynomial which vanishes on p. These are the same things as germs of rational functions near p.

The first thing is what is a rational function? It's a quotient of polynomials, where the denominator isn't vanishing where you are?

So what are germs? Germs are [lost]

Say X is a topological space or a manifold of any smooth or analytic kind you might like. So for a topological space you can look at the ring of continuous functions on an open set. In general you'll get the ring of morphisms on an open set.

Near a point, what is the germ of a function near $p \in X$? So S_p is the germs of functions at p. Different functions are defined on different sets. Near a point, they are all defined nearby. So elements in S_p will be $\{f \in S(U) : U \text{ is open in } X \text{ and contains } p\}/\sim$. These are denoted (f, U). And we have that $(f, U) \sim (g, V)$ if there exists open W containing p with $W \subset U \cap V$ with $f|_W = g|_W$. This is an equivalence relation, please check this.

So it comes with two operations, addition and multiplication. Add by restricting to some common ground and adding there. I claim that this is a local ring. The maximal ideal is functions that vanish at p. Evaluate any germ at p. The things in the kernel vanish at p. This is because the image is the ground field. Most geometric rings that come from nature are of this kind. So R_{m_p} is also a ring of germs. How?

So $f/g = f'/g'$ if there is an h such that $h(fg' - gf') = 0 \mod I$ for an h with $h(p) \neq 0$. Outside the hypersurface $h = 0$, we must have $fg' = gf'$ They agree as functions on X outside the closed set $h = 0 \cup g = 0 \cup g' = 0$. Conversely, if two things coincide in the Zariski topology you can reduce yourself to this by shrinking a little bit.

So this ring is the ring of germs of rational functions in the Zariski topology at p.

It's nice if this is maximal. If this is not maximal but prime you define a subvariety at X. These will be germs of functions defined on a Zariski neighborhood of that subvariety. But they're allowed to change from one local place to another. They agree along Y but they are defined on some neighborhood of Y.

That's important because you can realize R_p as a ring of germs. I don't know how convincing my last argument was. I proved almost all of it. I only showed in one direction.

This is why fractions are important, because they model germs. Now sheaves put together all of these as you vary p. in a coherent way.

Let me say what I plan to do next time. There are two main things I have to say. Why do you learn localization? Local rings are roughly the simplest rings after fields. Fields have no interesting theory. So to use a local ring you ignore everything except a point.

Say you have M an R-module. Everything, everyday, in algebraic geometry uses this. I want to show $M = 0$. That's equivalent to saying that $M_p = 0$ for all prime ideals, which is equivalent to saying that $M_m = 0$ for maximal ideals. Once I know that localization is flat I can do things after localizing.

$M \to^{\varphi} N$ is iso/mono/epi if and only if $M_p \to^{\varphi_p} N_p$ is.

This proves the Chinese remainder lemma in a second.

Let $p_i \cap p_j = 1$ for all $i \neq j$. Then we want to show $R \to \prod R/p_i$ is surjective. If I localize I get R_m and I get either R_p or 0, only the first if m contains p, with the map still the identity. So this is extremely powerful.

A morphism is 0 if the germ at every point is 0, and so on. Normally you come up with a formula to solve this. Concretely to solve, that's a good way. To argue about the morphism, that's simpler.

Every ideal is the intersection of the primes that contain it, that's one line from this. I'll be done next time with localization.