ALGEBRA III
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Let's try to finish the proof of that big theorem today. Hopefully next time we can play a bit with Macaulay. I'm going to summarize what I need.

Theorem 1. *Let M be an R-module.*

(1) *M has a finite composition series if and only if M is both Artinian and Noetherian.*
If M has a composition series M_i of length n then
(2) *Every chain of submodules has length at most n and can be refined to a composition series.*
(3) *$M = \bigoplus_{p\ maximal} M_p$, where p is the annihilator of $M_i/M_{i+1} = R/p$. Since $M_q = 0$ for q not such an annihilator there is no harm in doing this over all maximal ideals.*
The number of M_i/M_{i+1} isomorphic to R/p is $lt_{R_p}(M_p)$.
(4) *$M = M_p$ if and only if some power of p annihilates M.*

We proved most of this and I sketched a number of things. I've done the first two and sketched the beginning of the third part.

If $lt(M) = 1$ then $M = R/p$ with p maximal. Then either $Q = p$ so $M_Q = M$ or $Q \neq p$ and $M_Q = 0$. Also note that $(M_Q)_{Q'} = 0$ if $Q \neq Q'$ since you must go through the second of these.

So in the general case I have a finite composition series, and $lt(M_i/M_{i+1}) = 1$ because otherwise I could insert another module in my composition series. Then $(M_i/M_{i+1})_Q$ is either itself or is 0, as above. But since localization is exact this is $(M_i)_Q/(M_{i+1})_Q$.

This says that if Q is not one of the annihilator maximals, then $M_Q = 0$. If $Q \neq Q'$ maximals then by the same trick $(M_Q)_{Q'} = 0$.

Everything I've done so far we did last time. Now what is the map to the direct sum? It is the natural map, which takes $m \to (m/1, \ldots, m/1, \ldots)$. How do you show that something is an isomorphism? α is an isomorphism if and only if α_Q is an isomorphism for Q maximal. Once you localize to a prime you destroy all other primes and you have only a point. It may be fat geometrically, but you have reduced yourself to a neighborhood of that fat set. So roughly what you do is reduce yourself to a Zariski neighborhood. If you can do this at every big fat point then it works globally.

Remember that we can sum over all maximals, so I'm looking at the direct sum of the localization maps. The map is either 0 or an isomorphism. We get $M \to M_Q$ going to $M_Q \to^{id} M_Q$ and $(M \to M_p)_Q$ is $M_Q \to (M_p)_Q = 0$. It's a very simple proof once you use these techniques. Now you can actually do this by hand, using properties of the nilradical and so on. This is simpler.

Let's prove the last part. $M = M_p$ if and only if $p^N M = 0$. If $P^N M = 0$ then take any maximal $Q \neq p$, so that there are points in $p \backslash Q$. Then an element f of this set goes in M_Q to an invertible element. On the other hand, f^N multiplies the numerators to 0, so $M_Q = 0$. Since M is the direct sum of M_p, and all the other M_Q are killed in this way, this gives $M = M_p$ by the standard map.

Conversely is also easy. If $M = M_p$ then you localize at p. This tells you that $M_i/M_{i+1} = R/p$ for all i. Then $pM_i \in M_{i+1}$ So eventually if you shift n times you get into 0. You in fact get that $p^{lt(M)}$ annihilates M.

What is the consequence of all this? What do we know about Artinian rings now?

Localization is one of the most important tools in algebra, and it is easy.

Theorem 2. *Let R be a commutative ring with unit. The following are equivalent.*

(1) *R is Noetherian and all primes are maximal.*
(2) *R is finite length as an R-module.*
(3) *R is an Artinian ring.*

This will mean they are of Krull dimension zero. The first thing here is very geometric. The main theorem may have seemed dry, but it has lots of implications.

So Krull had the following idea. He essentially decided, knowing all about this theorem, that there was a notion of dimension in algebra. So in geometry what is the dimension of a topological space? How do we know the dimension of $\mathbb{R}^n$? It's not easy to show invariance of dimension. Again this boils down to what dimension means. I'm looking to a vector space, and I can look to a chain of vector spaces. It's the dimension of the longest chain of linear subspaces, nested. If I'm on an algebraic variety, I have subvarieties. So say I have $X \subset k^n$, algebraic varieties. I'm simplifying a bit. I look to chains of subvarieties $X = X_0 \supset X_1 \supset \cdots \supset X_n$, all of these subvarieties. Then the length of the longest chain you can achieve should be the dimension.

So what does this correspond to? $A(X) = k[x_1, \ldots, x_n]/I$, and the subvarieties correspond to prime ideals in the quotient. This gives me a chain $I = p_0 \subset p_1 \subset \cdots \subset p_n \subset k[x_1, \ldots, x_n]$. So p_n is maximal. So dimension corresponds to the longest chain of primes you can make between yourself and a maximal.

The correspondence with topology has to do with closed subspaces which are in some sense irreducible. If we knew about primary decomposition then we wouldn't need the ideal to be prime; we could write any ideal as an intersection of primes and take the maximal length of all of those.

So the maximal length of a chain of primes in a commutative (Noetherian) rings is the Krull dimension of R.

Here the Krull dimension is of R/I. We have $dim\ X = K - dim(A(X)$. The dimension of an ideal is the dimension of R/I.

This has a major disadvantage. If M is an R-module you can define the dimension of M to be $dim(R/ann(M))$. But the dimension of an ideal as an R-module is different than it is as an ideal, i.e., as specifying a quotient.

How do you call the Krull dimension of a ring where all prime ideals are maximal? It is zero. So Artinian rings are Noetherian rings of Krull dimension zero.

We'll prove this:

Lemma 1. *If R is Noetherian then dim $R[x] =$ dim $R + 1$.*

If R is not Noetherian then the dimension can increase by two or three, perhaps. I can point you to papers in the fifties.

Krull dimension 1 defines curves, Riemann surfaces. Dedekind rings are the bread and butter of number theory.

Krull dimension 2 are surfaces; we can still understand a lot. In higher dimensions it's still open.

There are other dimensions in algebra which are more algebraic that come from homological algebra.

Exercise 1. *The Krull dimension of $\mathbb{Z}$ is one; the Krull dimension of a field is zero.*

So let's prove this.

Keep in mind that when you have a chain of primes it's really a chain of subvarieties. It's based on the intuition that if you take a closed subset you maintain dimension. So let's assume R is Noetherian and all prime ideals are maximal, but R is not of finite length. I look to all ideals I such that R/I is not of finite length. Take the maximal such ideal with respect to this property. I can do this because this is nonempty, since R is of finite length, and R is Noetherian.

Now I claim that I is prime by contradiction, which will give me a contradiction to my hypothesis. So if not, there exist $a, b \in R$, with $ab \in I$ but neither of $a, b \in I$. So $0 \to R/(I : a) \to R/I \to R/I + (a) \to 0$ is exact.

Then $I + (a)$ strictly contains I so has finite length since I was maximal. Now, $b \in (I : a)$ but not in I so the other inclusion is also strict. This then contradicts the choice of I so that I is prime. Then I is maximal, so R/I is a field so it has finite length. So this proves that the first implies the second.

These are, let's say, slick proofs, but choose whichever you want.

Now let's prove the rest. Two implies three is clear from our theorem; since R has a finite composition series it is both Artinian and Noetherian.

So I assume R is Artinian; I will prove that R is of finite length, hence Noetherian, and then that primes are maximal.

So first this claim: The 0 ideal is the product of maximal ideals. Once I know this, the whole thing is over, I'll show you in a moment. How do you prove this? Really by hand.

Look to ideals J which are minimal ideals among those which are products of maximal ideals. You can choose a mimimal ideal since M is Artinian. This set is nonempty because it contains the maximal ideals. I want to show that $J = 0$. So $JM = J$. since $JM \subset J$ and J is minimal with respect to this property, for M maximal. This means that $J \subset M$, so that $J \in \cap M$, the Jacobsen radical.

How about J^2? This sits in J and is still a product of maximals, so $J^2 = J$. Assume $J \neq 0$. Choose a minimal ideal I among those such that $IJ \neq 0$. This is nonempty because it contains J. Then $(IJ)J = IJ$, so $IJ = I$ by minimality. Now some element f of I satisfies $fJ \neq 0$ Then $I = (f)$ by minimality. On the other hand, $IJ = I$. So $f = fg, g \in J$. This is $f(1-g) = 0$. So $g \in J$, so that g is in every maximal ideal; then $1-g$ is invertible, since a noninvertible element sits in a maximal ideal. If $(1-g)$ was in a maximal ideal, then 1 would be too. Then $f = 0$ so $I = 0$ so we have a contradiction, so $J = 0$.

So now how do we use this to show three implies two and one. Look to the product $m_1 \cdots m_t / m_1 \cdots m_{t+1}$. This is an R/m_{t+1} module. Since the quotient is a field this is a vector space. Each of these vector spaces, I claim, is of finite length; every chain of such submodules would be a chain of ideals of R, but R is Artinian so each of these has a finite composition series.

If you concatenate these then the last product is 0, and you get a finite length composition series. How do you prove that all prime ideals are maximal? If $p \subset R$ is a prime ideal then $p \ni 0$. So $p \supset m_1 \cdots m_s$ and so $p \supset m_i$ for some i.

This has a geometric interpetation. If $X \subset k^n$ an algebraic set. This is not equivalent, but it's a translation if R is an affine coordinate ring. The following are equivalent.

(1) X is a finite set
(2) $A(X)$ is a finite dimensional k-vector space whose dimension is the number of points.
(3) $A(X)$ is Artinian and its length is the number of points.

Let me make a comment why this is easy. Everything except one implies two is almost clear. What is $A(X)$? You can think of this as polynomials mod an ideal, or as restrictions of polynomials to a finite set. Then $A(X)$ is the product for the points in X of copies of k, since I can proscribe finitely many polynomial values. Then the dimension is exactly the number of points.

Why do you have three? An ascending or descending chain of modules is one of vector spaces. The length condition follows immediately.

The last thing is that if $A(X)$ is Artinian, then all prime ideals are maximal. The only subvarieties are those corresponding to maximal ideals, namely points. So X consists of finitely many points. So think of this as the geometric counterpoint. When you study Artinian rings you study finite sets of points.

Maybe next time we meet in the lab, I wanted to show you how to compute if something is Artinian, or how many solutions something has. Next time we meet in the basement; it will be rather short. Codes are essentially evaluating functions to points, and coding theory requires you to get a handle on this. Would it be interesting to look at this?