

ALGEBRA III
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I want to prove a very useful thing about modules. It will help us say more about Noetherian rings and Artinian rings. Artinian rings are very special, and Noetherian rings are the rings you play with if you are doing algebraic geometry.

Let M be an R -module (some of this can be done in the noncommutative world but again you have to be careful).

A chain of submodules is $M = M_0 \supset M_1 \supset \cdots \supset M_n$, where the inclusion is strict. This is a chain of length n . If these were vector spaces, then there would be a maximum length, bounded by the dimension. We would like something similar for modules.

The chains we're going to be interested in are called "composition series." These are chains which are maximal with respect to length.

You can always look to the quotient M_i/M_{i+1} . This is a very common thing to do. For it to be a composition series means that between these two I cannot introduce a module. Then there are only trivial submodules for M_i/M_{i+1} , i.e., this is simple.

This makes sense for infinite lengths but then I can't say very much, just, say, countable or uncountable, and that doesn't say much. So I'm interested in the cases where these have finite length.

Definition 1. $lt(M)$ is defined to be the least length of a composition series, or ∞ .

So this is kind of bad, taking a min of a max, so I want the composition series to have the same lengths. If these were vector spaces then the length would just be the dimension and these would all have the same length.

Now, what I would like to show is that the length of M is the length of any composition series. This is two statements. If the length is finite, then all are finite and of that number; if there is an infinite composition series then the length is infinite.

This theorem should go by two names, the Jordan-Holder theorem and the Chinese remainder lemma. You won't recognize either of them, but we will when we struggle.

Another observation is: N is a simple R module implies N is generated by a single element. Then $N = R/p$, where p is a maximal ideal (since any ideal containing p corresponds to a nonzero submodule of N). So $N = R/p$ is a field, which should rhyme with what happens with vector spaces. We can also write $p = \text{Ann}_R(N)$.

So $M_i/M_{i+1} \cong R/p_i$, p_i a maximal ideal equal to $\text{Ann}(M_i/M_{i+1})$. If you like, $p_i = (M_i : M_{i+1})$.

Theorem 1. *Jordan-Holder and Chinese remainder lemma* Let R be a commutative ring, M an R -module.

- (1) M has a finite composition series if and only if M is both artinian and noetherian as an R -module
- (2) If $M = M_0 \supset \cdots \supset M_n = (0)$ is a composition series of length n then
 - (a) Every chain of submodules has length at most n and can always be refined to a composition series, which then has length n .
 - (b) M is isomorphic to $\bigoplus M_p$, where p ranges over the annihilators of the quotients of successive terms in a composition series. Later I'll show that this is a unique collection of maximal ideals.
This is the Chinese remainder lemma. The number of quotients annihilated by a given p is $\text{lt}_{R_p}(M_p)$, and thus is independent of the chosen composition series.
 - (c) $M = M_p$ if and only if M is annihilated by a power of p .

Now, let me say a little bit about Artinian and Noetherian modules, and then let's prove this. We said what these mean in the case of rings, but let's make some definitions. Then we'll prove this.

Consider length as a substitute for dimension. The artinian condition is very strong.

Let me state an obvious lemma, I hope everyone will know a proof.

Lemma 1. *Let (Σ, \leq) be a poset. The following two are equivalent:*

- (1) Every increasing sequence $x_1 \leq x_2 \leq \cdots$ in Σ is stationary.
- (2) Every nonempty subset of Σ has a maximal element.

Do we need to prove this? How do you show that the first implies the second? So take an element of your subset. If you have a chain of length n with no maximal element, then there is a chain of length $n + 1$.

The other direction is trivial. Let your chain be your subset; then it has a maximal element so once it arrives there, it must be stationary.

So let Σ be the collection of submodules of an R -module M . We could use $\subseteq = \leq$. Then every sequence of weakly increasing submodules stabilizes (ascending chain condition) if and only if we have the equivalent condition, called the maximal condition of submodules. Such a module is Noetherian.

The other choice is $\supseteq = \geq$. So every weakly decreasing sequence of submodules stabilizes (descending chain condition) if and only if we have what is called the minimal condition on submodules.

Notice that we want to show that those modules with acc and dcc have finite composition series.

In the case of rings, Noetherian implies Artinian. These seem to be dual, but Artinian is much stronger. The Artinian condition makes things classifiable.

Exercise 1. *If V is a vector space over k then the following are equivalent:*

- (1) $\dim_k V < \infty$

- (2) $lt_k V < \infty$
- (3) V satisfies acc.
- (4) V satisfies dcc.

A couple of examples before we prove this, so we understand a little bit what these are.

Example 1. (1) Let M be a finite abelian group (\mathbb{Z} -module). Then M satisfies both acc., dcc.

- (2) $M = \mathbb{Z}$ Then M satisfies acc but not dcc because (2), (4), (8) is a strictly descending chain with no minimum.
- (3) $M \subset \mathbb{Q}/\mathbb{Z}$. Let M consist of the elements whose order is a power of a given prime p . Then the exercise which is easy is that there is exactly one subgroup M_n which has order p^n for all n . Then $M_0 \subset M_1 \subset \dots$ and these are not equivalent, so this does not satisfy acc. A descending chain must begin with a finite group so M satisfies dcc.
- (4) Look to $\{m/p^n, m \in \mathbb{Z}, n \in \mathbb{N}\} \subset \mathbb{Q}$. This satisfies neither acc nor dcc. More conceptually, it's easier to relate this to the previous examples. Call this H , which has all integers in it. Then there is a short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow H \rightarrow M \rightarrow 0$. So the middle term of a short exact sequence satisfies acc or dcc only if the other terms do.
- (5) Look at $k[x_1, \dots, x_n] = R = M$. This satisfies the acc but not dcc. Take the powers of a variable, for instance. A ring is called Noetherian (Artinian) if it is Noetherian (Artinian) as a module over itself.
- (6) Look at $k[x_1, \dots, x_n]$. This is neither Noetherian nor Artinian.
- (7) Let X be an infinite compact Hausdorff topological space, and let $R = M = \mathcal{C}(X)$. This is not Noetherian. Take a strictly descending sequence of closed sets, and look to functions which vanish on these sets. This is strict because compact Hausdorff implies normal.

So let me go back to the theorem. There is one extra thing that people use a lot. We've seen that something in the middle of a short exact sequence is Artinian/Noetherian if and only if the things at the ends are. Inductively, then,

Lemma 2. (1) If R is Noetherian(Artinian) then R^n is Noetherian(Artinian)
 (2) If R is Noetherian(Artinian) and M is finitely generated then M is Noetherian(Artinian)

The second comes from viewing M as a quotient of R^m .

Today it's mainly about Artinian modules.

Let's prove part one of our theorem. I would like to prove that if $N \subset M$ implies that $l(N) < l(M)$. How am I going to do it? If I take a composition series for M I have to trim it down to get one for N .

Say $M = M_0 \supset \dots M_n = 0$ is a composition series of the least possible length. I define $N_i = M_i \cap N$ for all i . If I look to N_{i-1}/N_i , this sits in M_{i-1}/M_i . This latter is simple, so the former is either 0 or the full module. If this is ever 0 then I have a collision of N_{i-1} with N_i , so I have to delete some terms. So a composition series for N will be obtained by deleting some elements of the composition series for M .

We cannot always have $N_{i-1}/N_i = M_{i-1}/M_i$ If you apply this for the last one, you have $N_{n-1}/0 = M_{n-1}/0$. This implies that $N_{n-1} = M_{n-1}$. Then by induction if all of these are the same then $N = M$.

Corollary 1. *As a consequence, any chain of submodules of M has length at most the length of M .*

If you had a sequence $M = M_0 \supset \cdots M_s$ then the length of M_1 is strictly smaller than the length of M_0 . So if s was larger than the length, then there would be a submodule with length less than 0.

Corollary 2. *A composition series has length at most $lt(M)$; on the other hand a composition series has length (by definition) $lt(M)$. So it has length exactly $l(M)$.*

In particular this implies the refinement condition.

If you have a finite composition series then every ascending or descending chain has to stop because otherwise the length would not be finite.

Corollary 3. *$l(m)$ is additive, i.e., if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ then $lt(M) = lt(M') + lt(M'')$.*

So it behaves like dimension. How do we prove this? Take a composition series for M'' and pull it back, you get a sequence which is maximal but begins with M'' . Now push forward a series for M' and they will line up.

Now let's take care of the direct sum condition. How do you localize a finite length module?

What is the simplest case? $lt(M) = 1$ means M has a single generator. So this looks like R/p , with p maximal. Choose Q a prime ideal and I would like to localize M at Q . What are the choices? Either $Q = p$, in which case $(R/p)_p$. Now everything in R/p other than 0 is invertible so this is R/p . Case two is $Q \neq p$ and we have $p \notin Q$. So look to $(R/p)_Q$. This is 0. This is the discussion we had when we gave the second proof of the Chinese remainder lemma. Then 0 is a denominator so the localization is 0.

Corollary 4. *Let Q, Q' be distinct prime ideals. Then $((M)_Q)_{Q'} = 0$. Then $M_Q = M$ or $M_Q = 0$ and in the first case $M_{Q'} = 0$.*

If $lt(M) < \infty$ then I can look to a composition series $M = M_0 \supset \cdots M_n = 0$. Then I can localize to get $M_Q = (M_0)_Q \supseteq \cdots \supseteq (M_n)_Q$. I know that $(M_i/M_{i+1})_Q = (M_i)_Q / (M_{i+1})_Q$. But $lt(M_i/M_{i+1}) = 1$ so this last quotient is either 0 or M_i/M_{i+1} if $Q = Ann(M_i/M_{i+1})$.

Then M_Q has a composition series whose length is the number of times that Q is an annihilator of M_i/M_{i+1} .

Just to say a word, you have a map taking M to $M/1$ in every localization. Once I localize to a maximal ideal, the only thing that survives is when $p = Q$. So you use the full force of what we learned last time. I'm five lines before the proof and then we'll see applications. Once you see those, Tuesday we will probably meet again downstairs to play a bit with Macaulay.